W-visible submodules and W-fully visible modules

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DOI: <https://doi.org/10.52866/ijcsm.2019.01.01.005>

Received December 2019; Accepted December 2019; Available online January 2020

ABSTRACT: In this paper, we presented a special type of submodule named W-visible submodule, which is weaker than the visible submodule, where a proper submodule W of a T -module X is said to be W-visible if $W = IW$ for some non-zero proper ideal I of T . Also, we presented the concept of W-fully visible module, where a module X over a ring T is said to be W-fully visible if any proper submodule of X is W-visible submodule. This concept is considered weaker than the concept of fully visible module. In this study, we achieved numerous results and characteristics that belong to this concept.

Keywords: Visible submodule; W-visible submodule; Fully visible module; W-fully visible module.

1. INTRODUCTION

Let X be a unitary module and T be a commutative ring. In [1], Mahmood and Buthyna submitted a concept of visible submodule which is defined as a proper submodule W of a module X over a ring T to achieve $W = IW$ for every non-zero ideal I of T . Also, Mahmood [2] presented the concept of fully visible module, which is defined as the module in which all submodules are visible.

In section two of this paper, a special type of submodule that is weaker than the visible submodule was presented; this submodule has been called the W-visible submodule. In this submodule, a proper submodule W of a T -module X is said to be W-visible if $W = IW$ for some non-zero proper ideal I of T .

It is clear that every visible submodule is W-visible, but the converse is not true, and we have been able to give an example of this.

In section three, we presented the concept of W-fully visible module, which is a module filled with W-visible submodules; in other words, a module X over a ring T is said to be W-fully visible if any proper submodule of X is W-visible submodule. This concept is considered weaker than the concept of fully visible module.

Many characterizations of W-visible submodules and W-fully visible modules have been submitted to many different properties.

2. W-VISIBLE SUBMODULES AND SOME OF THE PROPERTIES

Here, we introduce the concept of W -visible submodules which is considered weaker than the concept of visible submodules. Several results have been proven and the advantages have been given to describe this definition.

Definition 2.1. A proper submodule W of an T -module X is called W -visible if there exists a non-zero proper ideal C of T , such that $W = CW$.

Remarks and Examples 2.1.

- 1) A zero submodule of some T -module X can be W -visible, but in others, a zero submodule can be non- W -visible. In particular, Z_P as a Z_P -module, P is a prime number and Q as a Q -module has a zero submodule and cannot be W -visible.
- 2) A submodule $\begin{pmatrix} \bar{2} \end{pmatrix}$ of a Z -module Z_4 is W -visible since there exists a non-zero proper ideal $C = (3)$ of Z such that $\begin{pmatrix} \bar{2} \end{pmatrix} = (3) \begin{pmatrix} \bar{2} \end{pmatrix}$. Also, the submodules $\begin{pmatrix} \bar{2} \end{pmatrix}$ and $\begin{pmatrix} \bar{3} \end{pmatrix}$ in a module Z_6 on a ring Z are W -visible since $\begin{pmatrix} \bar{2} \end{pmatrix} = (2) \begin{pmatrix} \bar{2} \end{pmatrix}$ and $\begin{pmatrix} \bar{3} \end{pmatrix} = (3) \begin{pmatrix} \bar{3} \end{pmatrix}$ for some ideals (2) and (3) of Z , respectively.
- 3) In a more general form, any proper submodule of a Z -module Z_n , n , is an integer and can be W -visible.
- 4) All proper submodules, except the zero submodule of a module Z_n on a ring Z_n , n , are an integer and not prime; $n > 1$ cannot be W -visible.
- 5) In a module Z_{12} as Z_{36} -module, there are proper submodules represented by $\begin{pmatrix} \bar{3} \end{pmatrix}$, $\begin{pmatrix} \bar{4} \end{pmatrix}$ and $\begin{pmatrix} \bar{6} \end{pmatrix}$ be W -visible. Since there are $\begin{pmatrix} \bar{9} \end{pmatrix}$ and $\begin{pmatrix} \bar{4} \end{pmatrix}$, they can be ideals of Z_{36} such that $\begin{pmatrix} \bar{3} \end{pmatrix} = (\bar{9}) \begin{pmatrix} \bar{3} \end{pmatrix}$, $\begin{pmatrix} \bar{4} \end{pmatrix} = (\bar{4}) \begin{pmatrix} \bar{4} \end{pmatrix}$ and $\begin{pmatrix} \bar{6} \end{pmatrix} = (\bar{9}) \begin{pmatrix} \bar{6} \end{pmatrix}$. At the same module Z_{12} , there is an ideal $\begin{pmatrix} \bar{2} \end{pmatrix}$ that could not be W -visible since $\begin{pmatrix} \bar{2} \end{pmatrix}$, $(\bar{4})$, $\begin{pmatrix} \bar{6} \end{pmatrix}$, $(\bar{9})$ and $(\bar{12})$ are non-zero proper ideals of Z_{36} . But,

$$\begin{pmatrix} \bar{2} \end{pmatrix} \begin{pmatrix} \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{4} \end{pmatrix} \neq \begin{pmatrix} \bar{2} \end{pmatrix}$$

$$\begin{pmatrix} \bar{4} \end{pmatrix} \begin{pmatrix} \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{8} \end{pmatrix} \neq \begin{pmatrix} \bar{2} \end{pmatrix}$$

$$\begin{pmatrix} \bar{6} \end{pmatrix} \begin{pmatrix} \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{0} \end{pmatrix} \neq \begin{pmatrix} \bar{2} \end{pmatrix}$$

$$\begin{pmatrix} \bar{9} \end{pmatrix} \begin{pmatrix} \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{6} \end{pmatrix} \neq \begin{pmatrix} \bar{2} \end{pmatrix}$$

$$\begin{pmatrix} \bar{12} \end{pmatrix} \begin{pmatrix} \bar{2} \end{pmatrix} = \begin{pmatrix} \bar{0} \end{pmatrix} \neq \begin{pmatrix} \bar{2} \end{pmatrix}$$
 That is, $\begin{pmatrix} \bar{2} \end{pmatrix}$ is not a W -visible submodule of Z_{12} .
- 6) The submodules $\begin{pmatrix} \bar{2} \end{pmatrix}$ and $\begin{pmatrix} \bar{4} \end{pmatrix}$ are not W -visible in a module Z_8 over a ring Z_{16} , because there is no ideal C of Z_{16} such that $\begin{pmatrix} \bar{2} \end{pmatrix} = C \begin{pmatrix} \bar{2} \end{pmatrix}$ and $\begin{pmatrix} \bar{4} \end{pmatrix} = C \begin{pmatrix} \bar{2} \end{pmatrix}$.
- 7) Every non-zero submodule W of a Z -module Z is not W -visible, since there is no non-zero proper ideal C of a ring Z such that $W = CW$.
- 8) Consider the module $Z \oplus Z_n$ as a Z -module; n is an integer, $n > 1$, all the proper submodules of $Z \oplus Z_n$ which are in the form of $(0) \oplus W$ and W are a submodule of Z_n and can be W -visible. Other proper submodules of $Z \oplus Z_n$ that are in the form $W_1 \oplus W_2$, where W_1 is a non-zero submodule of Z and W_2 is any submodule of Z_n , cannot be W -visible.

9) As for (8), if $n = 1$, then, all non-zero proper submodules of a module $Z \oplus Z$ over Z are not W -visible.

10) A visible submodule of X is always W -visible submodule, but the converse is not true in general, and the example below shows that:

The submodule $\begin{pmatrix} \bar{2} \\ 2 \end{pmatrix}$ of a Z -module Z_4 is W -visible by (2), but it is not visible because we cannot guarantee that $\begin{pmatrix} \bar{2} \\ 2 \end{pmatrix} = C \begin{pmatrix} \bar{2} \\ 2 \end{pmatrix}$ for all non-zero proper ideals of a ring Z .

11) Any proper submodule of W -visible submodule is also W -visible

Proof. (11):

Let K be a proper submodule of W -visible W ; then, there exists a non-zero proper ideal C of T such that $W = CW$, $K \subseteq W = CW$, implying that $K + CW = CW \dots (1)$

Also, from the same above inclusion, we get:

$CK \subseteq CW$, implying that $CW + CK = CW \dots (2)$

From (1) and (2), we get $CW + K = CW + CK$ and hence, $K = CK$.

Thus, K is a W -visible submodule of W .

The opposite of (11) is not always true; let us illustrate this with the example below:

(0) is a W -visible submodule of a Z -module Z by (1), and $(0) \subseteq (2)$, where (2) is a submodule of the module Z over a ring Z , but (2) is not W -visible by (7).

In number (12), we discuss the intersection of two W -visible submodules.

12) Let W_1 and W_2 be two submodules of X which at least one of them is W -visible T -submodule. Then, $W_1 \cap W_2$ is also W -visible T -submodule.

Proof. (12):

It is known that $W_1 \cap W_2 \subseteq W$; also, $W_1 \cap W_2 \subseteq W_2$. Then, by (11), we obtain $W_1 \cap W_2$ which is a W -visible submodule.

As a generalization of (12), we get:

13) If $\{W_i\}_{i=1}^n$ is a finite collection of submodules and W_i is a W -visible submodule for some i , then, $\bigcap_{i=1}^n W_i$ is also W -visible.

Proof. (13):

By induction on n ;

14) Let T be a field; then, every proper submodule of a T -module X is not W -visible.

Proof. (14):

Suppose K is a proper submodule of X ; since T is a field, then, T has only two ideals (0) and T , and hence, we can not find a non-zero ideal I of T such that $K = IK$. Therefore, K is not a W -visible submodule of X .

Proposition 2.1. Let X and N be two T -modules and $f : X \rightarrow N$ be a T -epimorphism.

1) If W is a W -visible submodule of X , then, $f(W)$ is a W -visible submodule of N .

2) If K is a W -visible submodule of N , then, $f^{-1}(K)$ is a W -visible submodule of X .

Proof. (1)

Let W be a proper W -visible submodule of X . Then, $W = IW$ for some non-zero proper ideal I of T . $f(W) = f(IW) = If(W)$. Therefore, $f(W)$ is a W -visible submodule of N .

The condition f is an epimorphism and is not necessary here since we can take $f : Z_4 \rightarrow Z_4$ such that $f(x) = 4x \forall x \in Z_4$, where Z_4 is a Z -module. It is easy to show that f is not epimorphic. Let $-, W$ is a W -visible submodule of Z_4 and $-$ is also a W -visible of Z_4 .

Proof. (2)

Let K be a W -visible submodule of N , then, there exists a non-zero proper ideal J of T such that $K = JK$; therefore, $f^{-1}(K) = f^{-1}(JK) = Jf^{-1}(K)$ and hence, $f^{-1}(K)$ is a W -visible submodule of X .

In the following, we are working characterizations for the W -visible submodule.

Proposition 2.2. Let X be a T -module and N be a proper submodule of X . Then, the following are equipollent:

1) N is W -visible.

2) $(IN : N) = T$ for some nonzero proper ideal I of T .

3) $(rN : N) = T$ for nonzero element $r \in T$, $r \neq 1$.

4) $N = rN$ for some nonzero element $r \in T$, $r \neq 1$.

Proof.

(1) $\implies$ (2) Suppose that N is W -visible, then, $IN = N$ for some non-zero proper ideal I of T and hence, $(IN : N) = (N : N) = T$. Therefore, (2) holds.

(2) $\implies$ (3) The proof is direct.

(3) $\implies$ (4) From (3), we get $N \subseteq rN$ for some non-zero $r \in T$, $r \neq 1$, also, we have $rN \subseteq N$. Therefore, $rN = N$.

(4) $\implies$ (1) Suppose (4) holds, then, $N = (r)N$ for some non-zero $r \in T$, $r \neq 1$. Therefore, take $I = (r)$ and hence, $N = IN$ which implies N is a W -visible submodule of X .

Proposition 2.3. Let X_1 and X_2 be two T -modules and $X = X_1 + X_2$. Let W be a proper submodule of X . If $W = K + L$ for some $K \leq X_1$, $L \leq X_2$ such that W is W -visible submodule. Then, each of K and L are W -visible submodules.

Proof. We have W as a W -visible submodule of X , then, $IW = W$ for some non-zero ideal I of T , but $W = K + L$, then, $I(K + L) = K + L$ and hence, $IK + IL = K + L$. Therefore, $IK = K$ and $IL = L$ which implies L, K are W -visible submodules of X_1, X_2 , respectively.

Remark 2.1. The opposite of Proposition (2.5) does not hold in general; since the Z -module $X = Z_{10} + Z_9$. If $W = \begin{pmatrix} 5 \\ \bar{5} \end{pmatrix} + \begin{pmatrix} 3 \\ \bar{3} \end{pmatrix}$, where $\begin{pmatrix} 5 \\ \bar{5} \end{pmatrix}$ and $\begin{pmatrix} 3 \\ \bar{3} \end{pmatrix}$ are W -visible submodules of Z_{10} and Z_9 , respectively by Remarks and Examples ((2.2),(2)), but W is not a W -visible submodule.

Next, we explain the behavior of W -visible submodules of a module X over T under localization.

Proposition 2.4. Let X be a T -module; then, a proper submodule N of X is W -visible if and only if N_P is W -visible submodule of a T_P -module X_P for every maximal ideal P of T .

Proof. First direction: Let N be a proper submodule of X . To prove that N is a W -visible submodule of X ; since N_P is a W -visible submodule of X_P , then, there exists a non-zero proper ideal I_P of T_P such that $N_P = I_P N_P$ which implies $N_P = (IN)_P$ for every maximal ideal P of T by [4] and hence, $N = IN$ for some non-zero proper ideals I of T . Therefore, N is a W -visible submodule of X .

Second direction: Suppose N is a W -visible submodule of X over T ; To prove N_P is a W -visible submodule of X_P over T_P (for every maximal ideal P of T), $N = IN$ for some non-zero proper ideals I of T which implies $N_P = (IN)_P$ (for every maximal ideal P of T). And by [4], we get $N_P = I_P N_P$ since I_P is a non-zero proper ideal of T_P . This leads to the proof that N_P is a W -visible submodule of X_P .

3. W-FULLY VISIBLE MODULES

In this section, the concept of W -fully visible has been introduced, where the concept of the fully visible module is stronger than the concept of W -fully visible and we were able to give an example in this regard.

Some characterizations and many properties of this concept are proven; also, the tensor product of two W -fully visible modules is studied by giving some important results that illustrate it.

In the following, we submit the concept of W -Fully visible modules.

Definition 3.1. Let X be a T -module; then, X is called W -fully visible if every proper submodule of X is W -visible.

Remarks and Examples 3.1.

1) Consider the module Z_n over a ring Z ; n is a positive integer and a W -fully visible module.

2) Z_n as a Z_n -module, n is a positive integer and $n \geq 1$ is not W -fully visible.

3) The module $Z \oplus Z_n$ over Z , n is a positive integer and $n \geq 1$ is not a W -fully visible module.

4) Z_{p^∞} as a Z -module is not a W -fully visible module. Since, every submodule of Z_{p^∞} has the form $(1/p^l + Z)$, l is a non-negative integer, then, there is no non-zero proper ideal J of Z such that $J(1/p^l + Z) = (1/p^l + Z)$.

- 5) All modules of fully visible are W -fully visible, but the opposite is not true in general, and the following example explains that: A module Z_4 on a ring Z is a W -fully visible module by number (1), but it is not a fully visible module by (2).
- 6) A module X over a field T is not a W -fully visible, especially for Q as a Q -module.

Proof. The result is directly from Remarks and Examples ((2.2), (14)).

7) Every simple T -module is W -fully visible

Proof. Suppose W is a proper submodule of X , but X is a simple module, then, (0) and X is the only submodule of X and hence, the submodule W must be zero (since W is a proper submodule).

8) A proper submodule of W -fully visible module is also W -fully visible. Note that module X over T is not necessary to be a W -fully visible module when any submodule of it is W -fully visible. Consider the following example: the module $Z \oplus Z_6$ as a Z -module is not a W -fully visible module while the submodule $(0) \oplus Z_6$ of it is a W -fully visible module, since all proper submodules $(0) \oplus (\bar{0})$, $(0) \oplus (\bar{2})$, $(0) \oplus (\bar{3})$ are W -visible submodules of $(0) \oplus Z_6$. Therefore, $(0) \oplus Z_6$ is a W -fully visible Z -module.

Results can be obtained from (8), note next:

- 9) Let X_1 and X_2 be two W -fully visible T -modules; then, $X_1 \cap X_2$ is also a W -fully visible module.
- 10) As a generalization of (9), let $\{X_i\}_{i=1}^n$ be a family of W -fully visible T -modules; then, $\bigcap_{i=1}^n X_i$ is a W -fully visible module.
- 11) A direct summand of W -fully visible module X is always W -fully visible.

Proposition 3.1. *The homomorphic image of W -fully visible is also W -fully visible.*

Proof. Let $f : \hat{X} \rightarrow \hat{N}$ be an epimorphism where $\hat{X}, \hat{N}$ are two T -modules. Suppose $\hat{X}$ is W -fully visible; let $\hat{W}$ be a proper submodule of $\hat{N}$. Then, $\hat{W} = f(W)$ for some proper submodule W of $\hat{X}$, but W is a W -visible submodule of $\hat{X}$; then, there exists a non-zero proper ideal C of T , implying that $W = CW$. Now, $\hat{W} = f(W) = f(CW) = Cf(W) = C\hat{W}$.

Therefore, $\hat{W}$ is a W -visible submodule of $\hat{N}$; thus, $\hat{N}$ is W -fully visible.

Proposition 3.2. *Let $\hat{X}_1$ and $\hat{X}_2$ be two T -module; if $\hat{X}_1 \simeq \hat{X}_2$, then, $\hat{X}_1$ is a W -fully visible module if and only if $\hat{X}_2$ is a W -fully visible module.*

Proof. Suppose $\hat{X}_1$ is a W -fully visible module; let $\hat{W}$ be a proper submodule of $\hat{X}_2$, and $T : \hat{X}_1 \rightarrow \hat{X}_2$ be isomorphism, then, there exist $W < \hat{X}_1$ such that $T(W) = \hat{W}$, but $\hat{X}_1$ is W -fully visible module, then, W is a W -visible submodule of $\hat{X}_1$, that is, $W = IW$ for some ideal I of T .

This implies that $\hat{W} = T(W) = T(IW) = IT(W) = I\hat{W}$; hence, $\hat{W}$ is a W -fully visible module of $\hat{X}_2$. Thus, $\hat{X}_2$ is a W -fully visible module.

Opposite direction : let W be a proper submodule of $\hat{X}_1$; then, $T(W)$ is a proper submodule of $\hat{X}_2$, implying the existence of a proper submodule $\hat{W}$ of $\hat{X}_2$ such that $T(W) = \hat{W} \dots (1)$.

But $\hat{W}$ is a W -visible submodule (since $\hat{X}_2$ is W -fully visible module);, therefore, there exists an ideal I of T , and implies that $\hat{W} = I\hat{W}$; hence, from (1), we get $T(W) = IT(W)$; this gives $T(W) = T(IW)$, but T is monomorphism. Therefore, $W = IW$, leading to W being a W -visible submodule and hence, $\hat{X}_1$ is a W -fully visible module.

Proposition 3.3. *Let X be a T -module and I is an ideal of T , such that $I \subseteq \text{ann}_T(X)$; then, X is a W -fully visible T -module if and only if X is a W -fully visible T/I -module.*

Proof. $\Rightarrow$) let X be a W -fully visible T -module. Then, for every proper submodule W of X , there exists a non-zero proper ideal J of T such that $W = JW$. Now, let $x \in W$, then, $x = \sum_{i=1}^n r_i a_i$ where $r_i \in J$ and $a_i \in W$, implying that $x = \sum_{i=1}^n (r_i + I) a_i$ where $r_i + I \in J/I$, $a_i \in W$.

Therefore, $x \in J/IW$, and $W \subseteq (J/I)W$. It is clear that $(J/I)W \subseteq W$ which implies that $W = (J/I)W$. Thus, W is a W -visible T/I -submodule of X ; this leads us to X being a W -fully visible T/I -module of X .

$\Leftarrow$) Suppose X is a W -fully visible T/I -module of X , then, for every proper submodule W of X , there exists a non-zero proper ideal J/I of T/I such that $W = J/IW$. Now, let $m \in W$; then, $m = \sum_{i=1}^n (s_i + I) b_i$ where $(s_i + I) \in J/I$ and $b_i \in W$, which implies that $m = \sum_{i=1}^n s_i b_i \in JW$ where $s_i \in J$ and $b_i \in W$ and hence, $W \subseteq JW$ and we have $JW \subseteq W$;,

therefore, $JW = W$. This means that W is a W -visible submodule of a T -module X and hence, X is a W -fully visible T -module.

Here, we will review the necessary and sufficient conditions for a module to be W -fully visible under the definition of tensor product.

Definition 3.2. : [3]

Let X_1 and X_2 be two T -modules; then, a tensor product of X_1 and X_2 is a pair (X, F) where X is a T -module and $F : X_1 \times X_2 \longrightarrow X$ is a bilinear map, such that for any T -module N and any bilinear map $g : X_1 \times X_2 \longrightarrow N$, there exist a unique homomorphism $l : X \longrightarrow N$, such that $loF = g$.

We demonstrate the following lemma to help us in giving our next major proposition.

Lemma 3.1. Let X be a module over a ring T and N is a proper submodule of X . Then, $N \otimes T/I \simeq N/IN$ for any ideal I of T .

Proof. Let $g : N \times T/I \longrightarrow N/IN$ such that $g(a, r+I) = ra + IN$

Where $a \in N$ and $r+I \in T/I$. It is easy to prove that g is well defined and bilinear mapping. Now, to show that g is an epimorphism; let $a+IN \in N/IN$, then, $g(a, 1+I) = a+IN$, where $(a, 1+I) \in N \otimes T/I$, that is, g is an epimorphism and hence, there exists a homomorphism $l : N \otimes T/I \longrightarrow N/IN$ such that $l(a \otimes (r+I)) = ra + IN$.

Where T is a bilinear mapping. Then, by the first fundamental theorem $\frac{N \otimes T/I}{Ker l} \simeq N/IN$;

$$\begin{aligned} Ker l &= \left\{ \sum_{i=1}^n a_i \otimes (r_i + I) \in N \otimes T/I ; l\left(\sum_{i=1}^n a_i \otimes (r_i + I)\right) = IN \right\} \\ &= \left\{ \sum_{i=1}^n a_i \otimes (r_i + I) \in N \otimes T/I ; l\left(\sum_{i=1}^n a_i \otimes r_i(1+I)\right) = IN \right\} \\ &= \left\{ \sum_{i=1}^n a_i \otimes (r_i + I) \in N \otimes T/I ; l\left(\sum_{i=1}^n r_i a_i \otimes (1+I)\right) = IN \right\} \\ &= \left\{ \sum_{i=1}^n r_i a_i + IN = 0 = IN \right\} \text{ hence } \sum_{i=1}^n r_i a_i \in IN. \end{aligned}$$

Therefore, $\sum_{i=1}^n r_i a_i = \sum_{j=1}^n s_j b_j$ where $s_j \in I$ and $b_j \in N$. Therefore, $\sum_{i=1}^n a_i \otimes (r_i + I) = \sum_{i=1}^n r_i a_i \otimes (1+I) = \sum_{j=1}^n s_j b_j \otimes (1+I)$

$$= \sum_{j=1}^n b_j \otimes (s_j + I) = \sum_{j=1}^n b_j \otimes 0 = 0.$$

Thus, $N \otimes T/I \simeq N/IN$.

The next theorem submits a characterization for a module to be a W -fully visible module under the tensor product.

Theorem 3.1. Let X be a module over a ring T and N be a proper submodule of X . Then, X is W -fully visible if and only if $ann(N \otimes T/I) = T$

Proof. Assume that X is a W -fully visible module and N be a proper submodule of X ; then, N is a W -fully visible submodule of X . We have from Lemma (3.7), $N \otimes T/I \simeq N/IN$ for all ideal I of T .

Hence, $ann(N \otimes T/I) = ann(N/IN) = (IN : N) = T$ by Proposition ((2.4), (2)).

Another side: Suppose that $ann(N \otimes T/I) = T$, we want to show that X is a W -fully visible module. Therefore, we must prove that all proper submodules of X are W -visible. Since N is a proper submodule of X and we have from Lemma (3.7), $N \otimes T/I \simeq N/IN$ for all ideal I of T ;

Therefore $T = ann(N \otimes T/I) = ann(N/IN) = (IN : N)$.

That is, $(IN : N) = T$ and by Proposition ((2.4), (2)), N is a W -visible submodule. Thus, X is a W -fully visible module.

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