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The New Strange Generalized Rayleigh Family: Characteristics and Applications to COVID-19 Data

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ABSTRACT: In this paper, we introduce a novel family of continuous distributions known as the Odd Generalized Rayleigh-G Family. Within this family, we present a special sub-model known as the odd Generalized Rayleigh Inverse Weibull (OGRIW) distribution. The OGRIW distribution is derived by combining the T-X family and the Generalized Rayleigh distribution. We provide a comprehensive expansion of the (PDF) and (CDF) for the OGRIW distribution. Additionally, we investigate several mathematical properties of the OGRIW distribution, including moments, moment-generating function, incomplete moments, quantile function, order statistics and Rényi entropy. To estimate the model parameters, we employ the maximum likelihood method, aiming to identify the parameter values that maximise the likelihood of the observed data.
Finally, we apply the proposed OGRIW distribution to two real COVID-19 datasets from Mexico and Canada. The results of these applications demonstrate that the new distribution exhibits remarkable flexibility and outperforms other comparative distributions in terms of accurately modelling the COVID-19 data.

Keywords: Inverse Weibull, moment, T-X family, quantile function and COVID-19 data.

1. INTRODUCTION

To accurately capture the behaviour of data in diverse situations, several approaches have been proposed to achieve more adaptable distributions. One way to achieve this is by introducing one or two parameters into the parent distributions, which introduces new concepts for flexible modelling in distribution theory. The number of shape parameters that characterise the cumulative distribution function (CDF) plays a vital role in ensuring the flexibility of distributions. Various methods have been developed to incorporate one or more parameters into the CDF of a distribution, thereby enhancing the flexibility and effectiveness of the resulting distribution for data modelling.

In addition, researchers have introduced established families of continuous probability distributions to enhance the flexibility of distributions for real data analysis. These families aim to provide more versatile distributions that can better represent and analyse real-world data. Some well-known families of distributions include the Generalized beta-G family by Alexander et al. [1], Gamma-G Type-3 by Torabi and Montazeri [2], Weibull-X family by Alzaatreh et al. [3], odd generalized exponential-G family by Tahir et al. [4], Kumaraswamy Weibull-generated-G family by Hassan and Elgarhy [5], generalized transmuted-G by Nofal et al. [6], a new Weibull-X family by Zubair Ahmad et al. [7] and Exponentiated T-X family by Alzaghal et al. [8].

The new family introduced in this paper is derived based on the Generalized Rayleigh (GR) distribution, which was initially introduced by Surles and Padgett [9] and later extended by Bhat and Ahmad [10]. The CDF of the GR distribution is a key component in the derivation of this new family.
The CDF of the GR distribution is expressed as:

$$F(x) = \left(1 - e^{-bx^2}\right)^c \quad , \qquad c, b > 0\,, x \geq 0 \tag{1}$$

And the expression for the probability density function (PDF) corresponding to equation (1) can be written as follows:

$$f(x) = 2cbxe^{-bx^2}\left(1 - e^{-bx^2}\right)^{c-1} \quad , c, b > 0\,, x \geq 0$$

Eugene et al. created the so-called family of beta-generated distributions by using the beta distribution as a generator. A beta-G random variable X's CDF is defined as follows: [11]

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$$G(X) = \int_{0}^{F(x;\xi)} r(t)\,dt$$

Where r(t) is the beta random variable's PDF. and $F(x;\xi)$ is the CDF of any random variable and X's range is [0,1].

Further, Torabi and Montazeri introduced Gamma-G Type-3, a new flexible family of distributions. The CDF of a Gamma-G Type-3 random variable X is defined as: [12]

$$G(X) = \int_{0}^{\frac{F(x;\xi)}{1-F(x;\xi)}} r(t)\,dt$$

Where the range of X is $(0,\infty)$.

We introduced a new method of generating Families in this study in the following way:
Multiply the CDF of a random variable X into beta –G [11] by the CDF of a random variable X into Gamma-G Type-3[12]. The result, denoted as $W(F(x,\zeta))$, is the new CDF of a random variable X.

$$W(F(x,\zeta)) = \frac{F(x,\zeta)}{1-F(x,\zeta)} * F(x,\zeta)$$

Where the range of x is $[0,\infty)$, and $W(F(x,\zeta))$ satisfies the following conditions:
i. $W(F(x;\xi)) \in [e,d], -\infty < e < d < \infty,$
ii. $W(F(x;\xi))$ is differentiable and monotonically non-decreasing.
iii. $W(F(x;\xi)) \to e, as\ x \to -\infty, and\ W(F(x;\xi)) \to d, as\ x \to \infty$
With the method Alzaatreh et al. [3], we define the CDF of the new Odd Generalized Rayleigh (OGR) Family:

$$G(x,\zeta) = \int_{0}^{W(F(x,\zeta))} 2cbt e^{-bt^2}\left(1-e^{-bt^2}\right)^{c-1} dt$$

The CDF of OGR-G is:

$$G(x,\zeta)_{OGR-G} = \left[1-e^{-b\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2}\right]^c \qquad (2)$$

The PDF associated with equation (2) can be expressed as follows:

$$G(x,\zeta)_{OGR-G} = 2cb[F(x,\zeta)]^3 f(x,\zeta)(2-F(x,\zeta))(1-F(x,\zeta))^{-3}\left[1-e^{-b\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2}\right]^{c-1} e^{-b\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2} \qquad (3)$$

Where $c, b > 0, x \geq 0$

2. The odd Generalized Rayleigh Inverse Weibull distribution (OGRIW)

Consider the Inverse Weibull (IW) distribution with parameters ρ (scale parameter) and θ (shape parameter). Then, the CDF and PDF of IW are given as follows:

$$F(x,\zeta)_{IW} = e^{-\rho x^{-\theta}} \qquad (4)$$

And

$$f(x,\zeta)_{IW} = \rho\theta x^{-(\theta+1)} e^{-\rho x^{-\theta}} \quad ,\rho,\theta > 0, x > 0 \qquad (5)$$

we introduce a new distribution called the odd Generalized Rayleigh Inverse Weibull (OGRIW) distribution by substituting equation (4) into equation (2). This results in:

$$G(x)_{OGRIW} = \left[1 - e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^c \quad (6)$$

And the corresponding PDF is as given by equation (3)

$$g(x)_{OGRIW} = 2cb\rho\theta\, e^{-4\rho x^{-\theta}} x^{-(\theta+1)} \left(2 - e^{-\rho x^{-\theta}}\right)\left(1 - e^{-\rho x^{-\theta}}\right)^{-3} \left[1 - e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c-1} e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2} \quad (7)$$

Where $c, b, \rho, \theta > 0, x \geq 0$.

For OGRIW distribution's hazard, as mentioned below, we derive equation (8):

$$h(x)_{OGRIW} = \frac{g(x)_{OGRIW}}{1 - G(x)_{OGRIW}}$$

$$h(x)_{OGRIW} = \frac{2cb\rho\theta e^{-4\rho x^{-\theta}} x^{-(\theta+1)} \left(2 - e^{-\rho x^{-\theta}}\right)\left(1 - e^{-\rho x^{-\theta}}\right)^{-3} \left[1 - e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c-1} e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}}{1 - \left[1 - e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^c} \quad (8)$$

Figures a and b display the PDF and h(x) plots of the OGRIW distribution for the specified values of the $c, b, \rho, and\ \theta$ and functions. As illustrated in Figure 1, the PDF exhibits right skewness.

a b

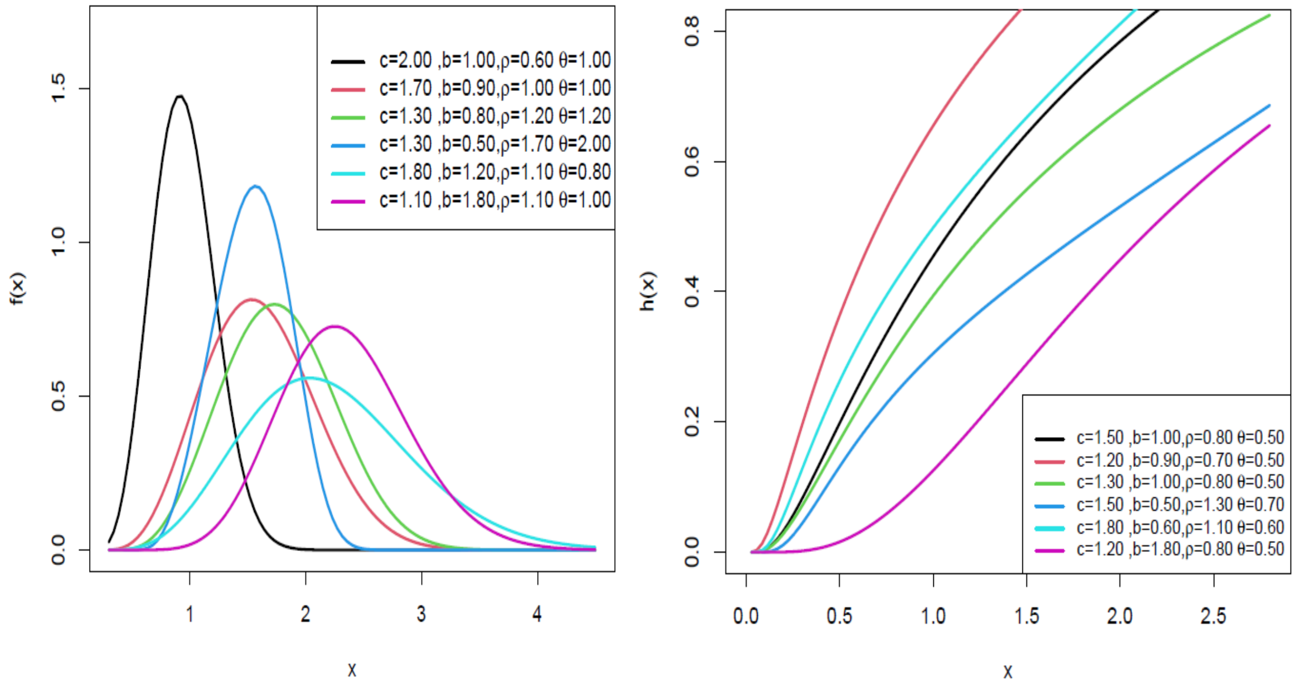

FIGURE 1.- (a) Plot of the PDF and (b) plot of the h(x) of the OGRIW distribution.

1. Mathematical Properties

This section explores the mathematical properties of the OGRIW distribution, including the useful Expansion of PDF and CDF, Moment, Moment Generating Function, Incomplete moment, Quantile function, Order Statistics and Rényi Entropy.

3.1 Useful Expansion

In this subsection, we simplify equation (3) by using the Exponential expansion $e^{-a} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} a^k$ and binomial series expansion $[1-u]^a = \sum_{i=0}^{\infty} (-1)^i \binom{a}{i} u^i$, $[1-u]^{-a} = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{j!\Gamma(a)} u^j \ : |u| < 1, a > 0$.[13-15]

By binomial series expansion, we derive:

$$\left[1 - e^{-b\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2}\right]^{c-1} = \sum_{i=0}^{\infty} (-1)^i \binom{c-1}{i} e^{-ib\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2}$$

And by using the Exponential expansion, we derive:

$$e^{-b(i+1)\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2} = \sum_{h=0}^{\infty} \frac{(-1)^h}{h!} b^h (i+1)^h \left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^{2h}$$

Then

$$g(x)_{OGR-G} = \sum_{i=h=0}^{\infty} \frac{(-1)^{i+h} b^h (i+1)^h}{h!} \binom{c-1}{i} 2cbf(x,\zeta)(2-F(x,\zeta))(F(x,\zeta))^{3+4h} (1-F(x,\zeta))^{-(3+2h)} \quad ,C > 1$$

Again, upon using binomial series expansion, we derive:

$$(1-F(x,\zeta))^{-(3+2h)} = \sum_{j=0}^{\infty} \frac{\Gamma(3+2h+j)}{j!\,\Gamma(3+2h)} [F(x,\zeta)]^j$$

Then

$$g(x)_{OGR-G} = \sum_{i=h=j=0}^{\infty} \frac{(-1)^{i+h} b^h (i+1)^h \Gamma(3+2h+j)}{h!\,j!\,\Gamma(3+2h)} \binom{c-1}{i} 2cbf(x,\zeta)(2-F(x,\zeta))(F(x,\zeta))^{3+4h+j}$$

We can expand $(F(x,\zeta))^{3+4h+j}$ to derive:

$$g(x)_{OGR-G} = 2₩f(x,\zeta)(F(x,\zeta))^q - ₩f(x,\zeta)(F(x,\zeta))^{q+1} \tag{9}$$

Where

$$₩ = \sum_{i=h=j=p=q=0}^{\infty} \frac{(-1)^{i+h+p+q} 2cb^{h+1}(i+1)^h \Gamma(3+2h+j)}{h!\,j!\,\Gamma(3+2h)} \binom{c-1}{i} \binom{3+4h+j}{p} \binom{q}{p}$$

By applying equation (4) and equation (5) to equation (9), we get equation (10):

$$g(x)_{OGRIW} = ₩2\rho\theta x^{-(\theta+1)} e^{-\rho(q+1)x^{-\theta}} - ₩\rho\theta x^{-(\theta+1)} e^{-\rho(q+2)x^{-\theta}} \tag{10}$$

Now, we simplify equation (2) by using the Exponential expansion and binomial series expansion:

$$\left[1 - e^{-b\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2}\right]^{c} = \sum_{r=0}^{\infty} (-1)^r \binom{c}{r} e^{-br\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2}$$

And

$$e^{-br\left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} b^m r^m \left(\frac{[F(x,\zeta)]^2}{1-F(x,\zeta)}\right)^{2m}$$

Then

$$G(x,\zeta)_{OGR-G} = \sum_{r=m=0}^{\infty} \frac{(-1)^{r+m} b^m r^m}{m!} \binom{c}{r} (F(x,\zeta))^{4m} (1-F(x,\zeta))^{-2m}$$

And

$$(1-F(x,\zeta))^{-2m} = \sum_{z=0}^{\infty} \frac{\Gamma(2m+z)}{z!\,\Gamma(2m)} [F(x,\zeta)]^z$$

Then

$$G(x,\zeta)_{OGR-G} = \sum_{r=m=z=0}^{\infty} \frac{(-1)^{r+m} b^m r^m \Gamma(2m+z)}{m!\, z!\, \Gamma(2m)} \binom{c}{r} \left(F(x,\zeta)\right)^{4m+z} \quad (11)$$

By applying equation (4) to equation (11), we get equation (12)

$$G(x)_{OGRIW} = \sum_{r=m=z=0}^{\infty} \frac{(-1)^{r+m} b^m r^m \Gamma(2m+z)}{m!\, z!\, \Gamma(2m)} \binom{c}{r} e^{-\rho(4m+z)x^{-\theta}} \quad (12)$$

Equations (11) and (12) play an important role in calculating the statistical properties of the distribution, including moment and incomplete moments.

3.2 Moment, Skewness and Kurtosis

The nth moment of X can be obtained from equation (10):[16]

$$\acute{\mu}_r = E(X^r)_{OGRIW} = \int_0^{\infty} x^r g(x)\, dx$$

$$\acute{\mu}_r = E(X^r)_{OGRIW} = ₩2\rho\theta \int_0^{\infty} x^r x^{-(\theta+1)} e^{-\rho(q+1)x^{-\theta}} dx - ₩\rho\theta \int_0^{\infty} x^r x^{-(\theta+1)} e^{-\rho(q+2)x^{-\theta}} dx$$

Let

$$y = \rho(q+1)x^{-\theta} \Rightarrow x = \left(\frac{y}{\rho(q+1)}\right)^{-\frac{1}{\theta}}, \qquad dy = -\theta\rho(q+1)x^{-(\theta+1)}dx \Rightarrow dx = -\frac{dy}{\theta\rho(q+1)x^{-(\theta+1)}}$$

Then

$$\acute{\mu}_r = E(X^r)_{OGRIW} = -\frac{2₩\rho\theta\Gamma\left(1-\frac{r}{\theta}\right)}{\theta\left(\rho(q+1)\right)^{1-\frac{r}{\theta}}} + \frac{₩\rho\theta\Gamma\left(1-\frac{r}{\theta}\right)}{\theta\left(\rho(q+2)\right)^{1-\frac{r}{\theta}}} \quad (13)$$

By utilising equation (13), we can determine various statistical moments of the distribution, such as the first moment ($\acute{\mu}_1$), representing the mean, the second moment ($\acute{\mu}_2$), the third moment ($\acute{\mu}_3$) and the fourth moment ($\acute{\mu}_4$).

$$\acute{\mu}_1 = E(X^1)_{OGRIW} = -\frac{2₩\rho\theta\Gamma\left(1-\frac{1}{\theta}\right)}{\theta\left(\rho(q+1)\right)^{1-\frac{1}{\theta}}} + \frac{₩\rho\theta\Gamma\left(1-\frac{1}{\theta}\right)}{\theta\left(\rho(q+2)\right)^{1-\frac{1}{\theta}}} \quad (14)$$

$$\acute{\mu}_2 = E(X^2)_{OGRIW} = -\frac{2₩\rho\theta\Gamma\left(1-\frac{2}{\theta}\right)}{\theta\left(\rho(q+1)\right)^{1-\frac{2}{\theta}}} + \frac{₩\rho\theta\Gamma\left(1-\frac{2}{\theta}\right)}{\theta\left(\rho(q+2)\right)^{1-\frac{2}{\theta}}} \quad (15)$$

$$\acute{\mu}_3 = E(X^3)_{OGRIW} = -\frac{2₩\rho\theta\Gamma\left(1-\frac{3}{\theta}\right)}{\theta\left(\rho(q+1)\right)^{1-\frac{3}{\theta}}} + \frac{₩\rho\theta\Gamma\left(1-\frac{3}{\theta}\right)}{\theta\left(\rho(q+2)\right)^{1-\frac{3}{\theta}}} \quad (16)$$

$$\acute{\mu}_4 = E(X^4)_{OGRIW} = -\frac{2₩\rho\theta\Gamma\left(1-\frac{4}{\theta}\right)}{\theta\left(\rho(q+1)\right)^{1-\frac{4}{\theta}}} + \frac{₩\rho\theta\Gamma\left(1-\frac{4}{\theta}\right)}{\theta\left(\rho(q+2)\right)^{1-\frac{4}{\theta}}} \quad (17)$$

By utilising equations (14), (15), (16) and (17), we can calculate the variance $Var(X) = E(X^2) - (E(X))^2$, skewness $SK = \frac{\mu_3}{\mu_2^{(\frac{3}{2})}}$ and kurtosis $KU = \frac{\mu_4}{\mu_2^2}$.

Table 1. – Numerical value of $\acute{\mu}_1, \acute{\mu}_2, \acute{\mu}_3, \acute{\mu}_4$, $Var(X)$, SK and KU of the OGRIW distribution.

c	b	ρ	θ	$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$	$Var(X)$	SK	KU
Values of parameter				values of properties						
			0.8	5	26	148	82	1	1.116	0.121
		1.5	**0.9**	4.2	18	83	47	0.36	1.086	0.145
	3		**0.8**	5.4	31	189	113	1.84	1.095	0.117
		1.6	**0.9**	4.4	21	103	63	1.64	1.070	0.142
1			**0.8**	5.1	28	160	67	1.99	1.079	0.085
		1.5	**0.9**	4.2	19	89	39	1.36	1.074	0.108
	3.5		**0.8**	5.5	33	204	92	2.75	1.076	0.084
		1.6	**0.9**	4.6	22	110	52	0.84	1.066	0.107
			0.8	3.8	15	67	131	0.56	1.153	0.582
		1.5	**0.9**	3.3	11	41	73	0.11	1.123	0.603
	3		**0.8**	4.1	18	85	181	1.19	1.113	0.558
		1.6	**0.9**	3.5	13	51	97	0.75	1.088	0.573
2			**0.8**	3.9	16	72	106	0.79	1.125	0.414
	3.5	**1.5**	**0.9**	3.4	12	44	60	0.44	1.058	0.416
			0.8	4.3	19	92	146	0.51	1.110	0.404
		1.6	**0.9**	3.6	13	54	80	0.04	1.102	0.473

Table (1) reveals some interesting observations regarding the behaviour of moments, variance, skewness and kurtosis in the OGRIW distribution under different parameter settings.

Firstly, when the values of c, b and ρ are held constant, and the value of θ is increased, we observe that the moments, variance and skewness decrease. However, the kurtosis increases for the OGRIW distribution.

Secondly, when the values of c, b and θ are kept constant, and the value of ρ is increased, we find that the moments, variance and skewness increase. Conversely, the kurtosis decreases for the OGRIW distribution.

Thirdly, when the values of c, θ and ρ are fixed, and the value of b is increased, we notice that the moments, variance and skewness increase. However, the kurtosis decreases for the OGRIW distribution.

Lastly, when the values of b, ρ and θ are maintained constant, and the value of c is increased, we observe that the moments, variance and skewness decrease. In contrast, the kurtosis increases for the OGRIW distribution.

3.3 Moment Generating Function

The MGF is given as follows:

$$M_x(t)_{\text{OGRIW}} = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} g(x)\, dx$$

Upon substituting equation (10) into the MGF equation, we get equation (18)

$$M_x(t)_{\text{OGRIW}} = \sum_{l=0}^{\infty} \frac{t^l}{l!}\left(-\frac{2\text{₩}\rho\theta\Gamma\left(1-\frac{r}{\theta}\right)}{\theta\left(\rho(q+1)\right)^{1-\frac{r}{\theta}}} + \frac{\text{₩}\rho\theta\Gamma(1-\frac{r}{\theta})}{\theta(\rho(q+2))^{1-\frac{r}{\theta}}}\right) \tag{18}$$

3.4 Incomplete Moments

The r-th incomplete moment R of X, $\acute{\mu}_r(v)$is given by equation [17]

$$\acute{\mu}_r(v) = \int_{-\infty}^{v} x^r g(x)_{\text{OGRIW}}\, dx$$

Upon substituting equation (10) into the above equation, we get:

$$\acute{\mu}_r(v) = \text{₩}2\rho\theta \int_0^v x^r x^{-(\theta+1)} e^{-\rho(q+1)x^{-\theta}} dx - \text{₩}\rho\theta \int_0^u x^r x^{-(\theta+1)} e^{-\rho(q+2)x^{-\theta}} dx$$

Let $t = \rho(q+1)x^{-\theta}$ then

$$\acute{\mu}_r(v) = -\frac{2\text{₩}\rho\theta}{\theta\left(\rho(q+1)\right)^{1-\frac{r}{\theta}}} \int_0^{\rho v(q+1)} t^{-\frac{r}{\theta}} e^{-t} dt + \frac{\text{₩}\rho\theta}{\theta(\rho(q+2))^{1-\frac{r}{\theta}}} \int_0^{\rho v(q+2)} l^{-\frac{r}{\theta}} e^{-l} dl$$

By utilising the definition of the complete Gamma function, the $\acute{\mu}_r(v)$ will be as follows:

$$\dot{\mu}_r(v) = -\frac{2₩\rho\theta}{\theta\left(\rho(q+1)\right)^{1-\frac{r}{\theta}}}\gamma\left(1-\frac{r}{\theta},\rho v(q+1)\right) + \frac{₩\rho\theta}{\theta\left(\rho(q+2)\right)^{1-\frac{r}{\theta}}}\gamma\left(1-\frac{r}{\theta},\rho v(q+2)\right) \quad (19)$$

3.5 Quantile function

The inverse of equation (6) allows us to obtain the quantile function:

$$\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^c = u$$

Upon applying algebraic operations, we arrive at the following formula:

$$Q(u) = \left(-\frac{1}{\rho}ln\left(\frac{-\left(-\frac{1}{b}ln\left(1-u^{\frac{1}{c}}\right)\right)^{\frac{1}{2}} \mp \sqrt{\left(-\frac{1}{b}ln\left(1-u^{\frac{1}{c}}\right)\right)^{\frac{1}{2}}+4\left(-\frac{1}{b}ln\left(1-u^{\frac{1}{c}}\right)\right)^{\frac{1}{2}}}}{2}\right)\right)^{-\frac{1}{\theta}} \quad (20)$$

3.6 Order Statistics

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ of size n from OGRIW distribution. The probability density function for order statistics is given as follows:[18]

$$g_{p:n}(x) = \frac{n!}{(p-1)!\,(n-p)!}[G(x)_{OGRIW}]^{p-1}[1-G(x)_{OGRIW}]^{n-p}g(x)_{OGRIW}$$

Substituting equations (6) and (7) into the $g_{p:n}(x)$ equation, we get equation (21) as follows:

$$g_{p:n}(x) = \frac{n!}{(p-1)!\,(n-p)!}\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c(p-1)}\left[1-\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^c\right]^{n-p}\left(2cb\rho\theta\, e^{-4\rho x^{-\theta}}x^{-(\theta+1)}\left(2 - e^{-\rho x^{-\theta}}\right)\left(1-e^{-\rho x^{-\theta}}\right)^{-3}\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c-1} e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right) \quad (21)$$

The PDF of the minimum order statistic, when p = 1, and the maximum order statistic, when p = n, of the OGRIW distribution, are respectively expressed as:

$$g_{1:n}(x) = n\left[1-\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^c\right]^{p-1}\left(2cb\rho\theta\, e^{-4\rho x^{-\theta}}x^{-(\theta+1)}\left(2 - e^{-\rho x^{-\theta}}\right)\left(1-e^{-\rho x^{-\theta}}\right)^{-3}\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c-1} e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right) \quad (22)$$

$$g_{n:n}(x) = n\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c(p-1)}\left(2cb\rho\theta\, e^{-4\rho x^{-\theta}}x^{-(\theta+1)}\left(2 - e^{-\rho x^{-\theta}}\right)\left(1-e^{-\rho x^{-\theta}}\right)^{-3}\left[1-e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c-1} e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right) \quad (23)$$

3.7 Rényi Entropy

The Rényi entropy of a random variable is a measure of an uncertainty's variation. The Rényi entropy is defined as follows:

$$T_R(\eta) = \frac{1}{1-\eta} log \int_0^\infty g^{\eta}(x)_{OGRIW} dx \quad , \quad \eta > 0, \eta \neq 1$$

By using Equation (7), we get:

$$T_R(\eta) = \frac{1}{1-\eta} log \int_0^\infty \left(2cb\rho\theta e^{-4\rho x^{-\theta}} x^{-(\theta+1)} \left(2 - e^{-\rho x^{-\theta}}\right)\left(1 - e^{-\rho x^{-\theta}}\right)^{-3} \left[1 - e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2}\right]^{c-1} e^{-b\left(\frac{e^{-2\rho x^{-\theta}}}{1-e^{-\rho x^{-\theta}}}\right)^2} \right)^{\eta} dx \tag{24}$$

4. Maximum Likelihood Estimation

Let $x_1, \dots, x_n$ be a random sample of size n from the OGRIW class having parameters $c, b, \rho, and\ \theta$. Consider $\epsilon = (c, b, \rho, \theta)^T$. The log-likelihood (LL) function is given as:

$$l = nlog(2) + nlogc + nlogb + nlog\rho + nlog\theta - (\theta+1)\sum_{i=1}^{n} logx_i - 4\rho \sum_{i=1}^{n} x_i^{-\theta} + \sum_{i=1}^{n} log\left(2 - e^{-\rho x_i^{-\theta}}\right) - 3\sum_{i=1}^{n} log\left(1 - e^{-\rho x_i^{-\theta}}\right) + (c-1)\sum_{i=1}^{n} log\left(1 - e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}\right) - b\sum_{i=1}^{n}\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2 \tag{25}$$

We take the partial derivative of the log-likelihood function for the parameters $c, b, \rho\ and\ \theta$ to obtain the probability estimate for each parameter, as follows:

$$\frac{\partial(l)}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} log\left(1 - e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}\right) \tag{26}$$

$$\frac{\partial(l)}{\partial b} = \frac{n}{b} - \sum_{i=1}^{n}\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2 + (c-1)\sum_{i=1}^{n} \frac{b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2 e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}}{1 - e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}} \tag{27}$$

$$\frac{\partial(l)}{\partial\rho}=\frac{n}{\rho}-4\sum_{i=1}^{n}x_i^{-\theta}+\sum_{i=1}^{n}\frac{x_i^{-\theta}e^{-\rho x_i^{-\theta}}}{2-e^{-\rho x_i^{-\theta}}}-3\sum_{i=1}^{n}\frac{x_i^{-\theta}e^{-\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}+(c-1)\sum_{i=1}^{n}\frac{2bx_i^{-\theta}\left(e^{-2\rho x_i^{-\theta}}\right)^2\left(-2+e^{-\rho x_i^{-\theta}}\right)e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}}{\left(1-e^{-\rho x_i^{-\theta}}\right)^3\left(1-e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}\right)}+2b\sum_{i=1}^{n}\frac{x_i^{-\theta}\left(e^{-2\rho x_i^{-\theta}}\right)^2\left(-2+e^{-\rho x_i^{-\theta}}\right)}{\left(1-e^{-\rho x_i^{-\theta}}\right)^3} \tag{28}$$

$$\frac{\partial(l)}{\partial\theta}=\frac{n}{\theta}-4\rho\sum_{i=1}^{n}x_i^{-\theta}\,ln(x_i)+\sum_{i=1}^{n}\frac{\rho x_i^{-\theta}e^{-\rho x_i^{-\theta}}\,ln(x_i)}{2-e^{-\rho x_i^{-\theta}}}-3\sum_{i=1}^{n}\frac{\rho x_i^{-\theta}e^{-\rho x_i^{-\theta}}\,ln(x_i)}{1-e^{-\rho x_i^{-\theta}}}+2(c-1)\sum_{i=1}^{n}\frac{2b\rho x_i^{-\theta}\,ln(x_i)\left(e^{-2\rho x_i^{-\theta}}\right)^2\left(-2+e^{-\rho x_i^{-\theta}}\right)e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}}{\left(1-e^{-\rho x_i^{-\theta}}\right)^3\left(1-e^{-b\left(\frac{e^{-2\rho x_i^{-\theta}}}{1-e^{-\rho x_i^{-\theta}}}\right)^2}\right)}+2b\sum_{i=1}^{n}\frac{\rho x_i^{-\theta}\,ln(x_i)\left(e^{-2\rho x_i^{-\theta}}\right)^2\left(-2+e^{-\rho x_i^{-\theta}}\right)}{\left(1-e^{-\rho x_i^{-\theta}}\right)^3} \tag{29}$$

We obtain the estimate of the Maximum Likelihood Estimation by equating the earlier equations to zero: $\frac{\partial(l)}{\partial c}=\frac{\partial(l)}{\partial b}=\frac{\partial(l)}{\partial\rho}=\frac{\partial(l)}{\partial\theta}=0$. The usage of R was necessary because these equations cannot be resolved analytically.

5. Applications to COVID-19 data

In this section, we compare the OGRIW distribution with other competing models using real-world COVID-19 mortality rates data from Mexico and Canada. The results are presented in Table 2.

Table 2. - Comparative distributions.

Distributions	CDF
❖ **Truncated Exponentiated Exponential Inverse Weibull distribution (TEEIW) (New).**	$F(x) = \frac{\left(1 - e^{-ce^{-\rho x^{-\theta}}}\right)^{b}}{(1 - e^{-c})^{b}}$
❖ **Beta Inverse Weibull distribution (BeIW) [19].**	$F(x) = pb(e^{-\rho x^{-\theta}}, c, b)$
❖ **Kumaraswamy Inverse Weibull distribution (KuIW) [20].**	$F(x) = 1 - \left(1 - e^{-\rho c x^{-\theta}}\right)^{b}$
❖ **Exponential Generalized Inverse Weibull distribution (EGIW) [21].**	$F(x) = \left(1 - \left(1 - e^{-\rho x^{-\theta}}\right)^{c}\right)^{b}$
❖ **Weibull Inverse Weibull distribution (WeIW) (New).**	$F(x) = 1 - exp\left(-b^{-c}\left(-log\left(1 - e^{-\rho x^{-\theta}}\right)\right)^{c}\right)$
❖ **Rayleigh Inverse Weibull distribution (RIW) (New).**	$F(x) = exp\left(-\frac{c}{2}\left(-log\left(e^{-\rho x^{-\theta}}\right)\right)^{2}\right)$
❖ **Gompertz Inverse Weibull distribution (GoIW) (New).**	$F(x) = \left(1 - \left(1 - e^{-\rho x^{-\theta}}\right)^{r}\right)^{u}$ $1 - exp\left(\left(\frac{r}{u}\right)\left(1 - \left(1 - e^{-\rho x^{-\theta}}\right)^{-u}\right)\right)$
❖ **Burr type X distribution (BX) [22].**	$F(x) = \left(1 - exp(-(ax)^{2})\right)^{b}$
❖ **Rayleigh distribution (R) [10].**	$F(x) = 1 - \exp(-l\ x^{2})$
❖ **Inverse Weibull distribution (IW) [23].**	$F(x) = exp(-ax^{-b})$

The first dataset (I): Represents the COVID-19 mortality rate data for Mexico over 108 days, specifically recorded from March 4th to July 20, 2020. The data provides information on the rough mortality rate, indicating the number of deaths related to COVID-19 during that period [24].
(8.826, 6.105 ,10.383, 7.267 ,13.220, 6.015 ,10.855, 6.122 ,10.685, 10.035, 5.242 ,7.630 ,14.604 ,7.903 ,6.327 ,9.391 ,14.962 ,4.730 ,3.215 ,16.498, 11.665 ,9.284, 12.878, 6.656, 3.440 ,5.854, 8.813 ,10.043, 7.260, 5.985, 4.424 ,4.344 ,5.143 ,9.935 ,7.840 ,9.550 ,6.968 ,6.370 ,3.537 ,3.286 ,10.158, 8.108 ,6.697 ,7.151 ,6.560 ,2.988 ,3.336 ,6.814 ,8.325 ,7.854 ,8.551 ,3.228, 3.499 ,3.751 ,7.486 ,6.625 ,6.140 ,4.909 ,4.661 ,1.867 ,2.838 ,5.392, 12.042, 8.696 ,6.412 ,3.395 ,1.815 ,3.327 ,5.406 ,6.182 ,4.949 ,4.089 ,3.359 ,2.070, 3.298 ,5.317 ,5.442 ,4.557 ,4.292 ,2.500 ,6.535 ,4.648 ,4.697 ,5.459 ,4.120, 3.922 ,3.219, 1.402 ,2.438, 3.257 ,3.632, 3.233 ,3.027, 2.352 ,1.205 ,2.077, 3.778, 3.218, 2.926, 2.601, 2.065, 1.041, 1.800, 3.029, 2.058, 2.326, 2.506, 1.923).

Based on the values presented in Table 4, it is evident that the OGRIW distribution exhibits the minimum values for various information criteria, such as Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), and Hanan and Quinn Information Criteria (HQIC). Additionally, the OGRIW distribution demonstrates the highest P-value for the Kolmogorov-Smirnov (KS) test, as well as the lowest values for the Cramér-von Mises (W) and Anderson-Darling (A) tests, in comparison to the competing distributions.

These findings lead us to conclude that the OGRIW distribution provides the best fit for the first dataset. Furthermore, Figure 3 supports this conclusion, illustrating that our distribution outperforms the comparative distributions, further emphasising its superiority in accurately modelling the data.

In summary, based on the superior performance in terms of information criteria and goodness-of-fit tests, we can confidently conclude that the OGRIW distribution is the most suitable model for the first dataset.

Table 3. - Statistical description for Data (I).

Var	n	mean	Sd	median	Min	Max	SK	KU
x	108	5.76	3.25	5.19	1.04	16.5	0.97	0.61

Table 4. - Goodness-of-fit statistics and KS p-value for Data (I).

Distribution	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
OGRIW	$\hat{c}$: 0.4151 $\hat{b}$: 1.9208 $\hat{p}$: 0.7258 $\hat{\theta}$: 0.4512	265.99	539.99	540.38	550.72	544.34	0.0562	0.3117	0.0704	0.658
TEEIW	$\hat{c}$: 2.7752 $\hat{b}$: 0.4394 $\hat{p}$: 17.515 $\hat{\theta}$: 1.2447	273.62	555.46	555.85	566.19	559.81	0.1871	1.1763	0.0777	0.530
BeIW	$\hat{c}$: 2.9331 $\hat{b}$: 4.0901 $\hat{p}$: 3.3877 $\hat{\theta}$: 0.8624	269.95	547.94	548.33	558.67	552.29	0.1314	0.8078	0.0859	0.401
KuIW	$\hat{c}$: 2.6479 $\hat{b}$: 5.1677 $\hat{p}$: 3.0775 $\hat{\theta}$: 0.8712	268.37	544.75	545.14	555.48	549.10	0.1011	0.6023	0.0767	0.547
EGIW	$\hat{c}$: 3.3066 $\hat{b}$: 2.9851 $\hat{p}$: 3.5637 $\hat{\theta}$: 0.8433	271.25	550.51	550.90	561.24	554.86	0.1558	0.9665	0.0846	0.421
RIW	$\hat{c}$: 2.1374 $\hat{b}$: 2.8817 $\hat{p}$: 0.8442	277.32	560.65	560.88	568.70	563.91	0.2647	1.6858	0.0869	0.387
BX	$\hat{c}$: 0.1514 $\hat{b}$: 1.0022	269.23	542.46	542.57	547.82	544.64	0.1184	0.7624	0.0938	0.297
R	$\hat{c}$: 0.0229	269.22	540.46	540.49	543.14	541.54	0.1185	0.7627	0.0933	0.303
IW	$\hat{c}$: 8.9181 $\hat{b}$: 1.6913	277.33	558.65	558.77	564.02	560.83	0.2653	1.6899	0.0864	0.394

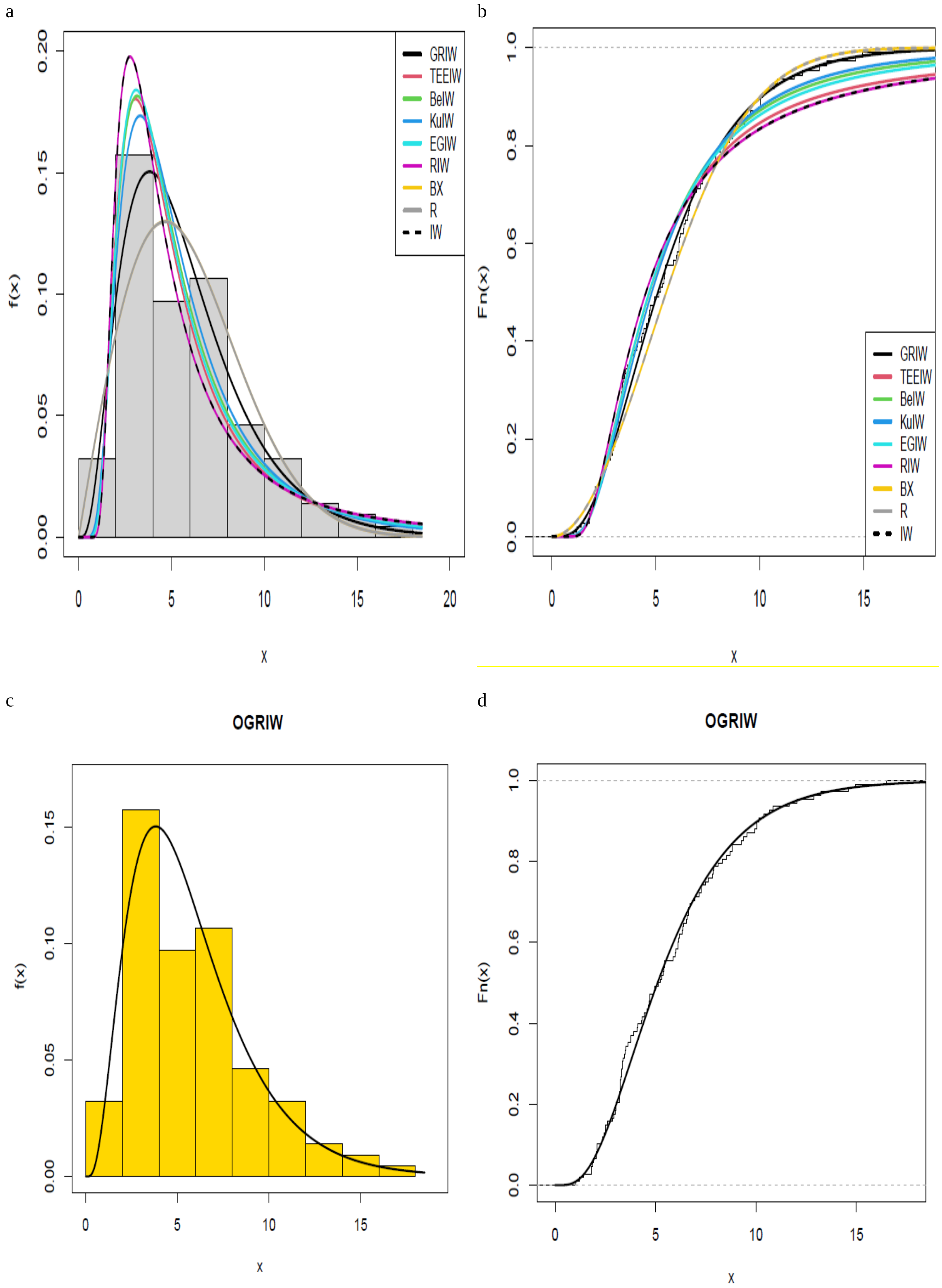

FIGURE 2.- (a) and (c) Estimated PDF and (b) and (d) CDF for data I.

The second dataset (II): Represents the COVID-19 mortality rate data for Canada over 36 days, specifically from April 10 to May 15, 2020. The data provides information on the rough mortality rate, indicating the number of deaths related to COVID-19 during that period. The data are as follows:[25]

(3.1091, 3.3825, 3.1444, 3.2135, 2.4946, 3.5146, 4.9274, 3.3769, 6.8686, 3.0914, 4.9378, 3.1091 ,3.2823, 3.8594, 4.0480, 4.1685, 3.6426, 3.2110, 2.8636, 3.2218, 2.9078, 3.6346, 2.7957, 4.2781, 4.2202, 1.5157, 2.6029, 3.3592, 2.8349, 3.1348, 2.5261, 1.5806, 2.7704, 2.1901, 2.4141, 1.9048).

Based on the values presented in Table 6, it is evident that the OGRIW distribution exhibits the minimum values for various information criteria, including AIC, CAIC, BIC, HQIC, as well as the KS, W and A tests, in comparison to the competing distributions. Additionally, the OGRIW distribution demonstrates the highest P-value for the KS test.

These observations lead us to conclude that the OGRIW distribution provides the best fit for the second dataset. Furthermore, Figure 3 further supports this conclusion, illustrating that our distribution outperforms the comparative distributions, reaffirming its superiority in accurately modelling the data.

In summary, based on the superior performance in terms of information criteria and goodness-of-fit tests, we can confidently conclude that the OGRIW distribution is the most suitable model for the second dataset.

Table 5. - Statistical description for Data (II).

Var	n	mean	Sd	median	Min	Max	SK	KU
x	36	3.28	1	3.18	1.52	6.87	1.16	2.81

Table 6. - Goodness-of-fit statistics and KS p-value for Data (II).

Distribution	MLEs	-2L	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value
OGRIW	$\hat{c}$: 0.1032 $\hat{b}$: 5.6016 $\hat{\rho}$: 0.3738 $\hat{\theta}$: 0.6856	48.11	104.23	105.52	110.56	106.44	0.0970	0.5674	0.1063	0.810
TEEIW	$\hat{c}$: 7.8910 $\hat{b}$: 0.4835 $\hat{\rho}$: 29.2929 $\hat{\theta}$: 1.9110	48.46	104.92	106.21	111.26	107.13	0.1112	0.6702	1.1342	0.535
BeIW	$\hat{c}$: 4.0700 $\hat{b}$: 5.5173 $\hat{\rho}$: 4.4957 $\hat{\theta}$: 1.4549	49.26	106.54	107.83	112.87	108.75	0.1398	0.8435	0.1382	0.496
KuIW	$\hat{c}$: 3.8275 $\hat{b}$: 5.0458 $\hat{\rho}$: 3.8863 $\hat{\theta}$: 1.7484	48.77	105.54	106.83	111.87	107.75	0.1237	0.7443	0.1334	0.542
EGIW	$\hat{c}$: 4.5081 $\hat{b}$: 3.9705 $\hat{\rho}$: 5.0300 $\hat{\theta}$: 1.3521	49.76	107.54	108.83	113.87	109.75	0.1548	0.9366	0.1419	0.463
WeIW	$\hat{c}$: 3.1630 $\hat{b}$: 5.3552 $\hat{\rho}$: 5.6274 $\hat{\theta}$: 5.7108	48.85	105.71	107.002	112.04	107.92	0.1303	0.7750	0.1280	0.596
GoIW	$\hat{c}$: 0.1733 $\hat{b}$: 1.3903 $\hat{\rho}$: 4.1515 $\hat{\theta}$: 2.2586	50.80	109.62	110.91	115.95	111.83	0.1567	0.8947	0.1338	0.539
RIW	$\hat{c}$: 2.8748 $\hat{b}$: 4.0349 $\hat{\rho}$: 1.5845	52.92	111.84	112.59	116.59	113.49	0.2548	1.5282	0.1737	0.227

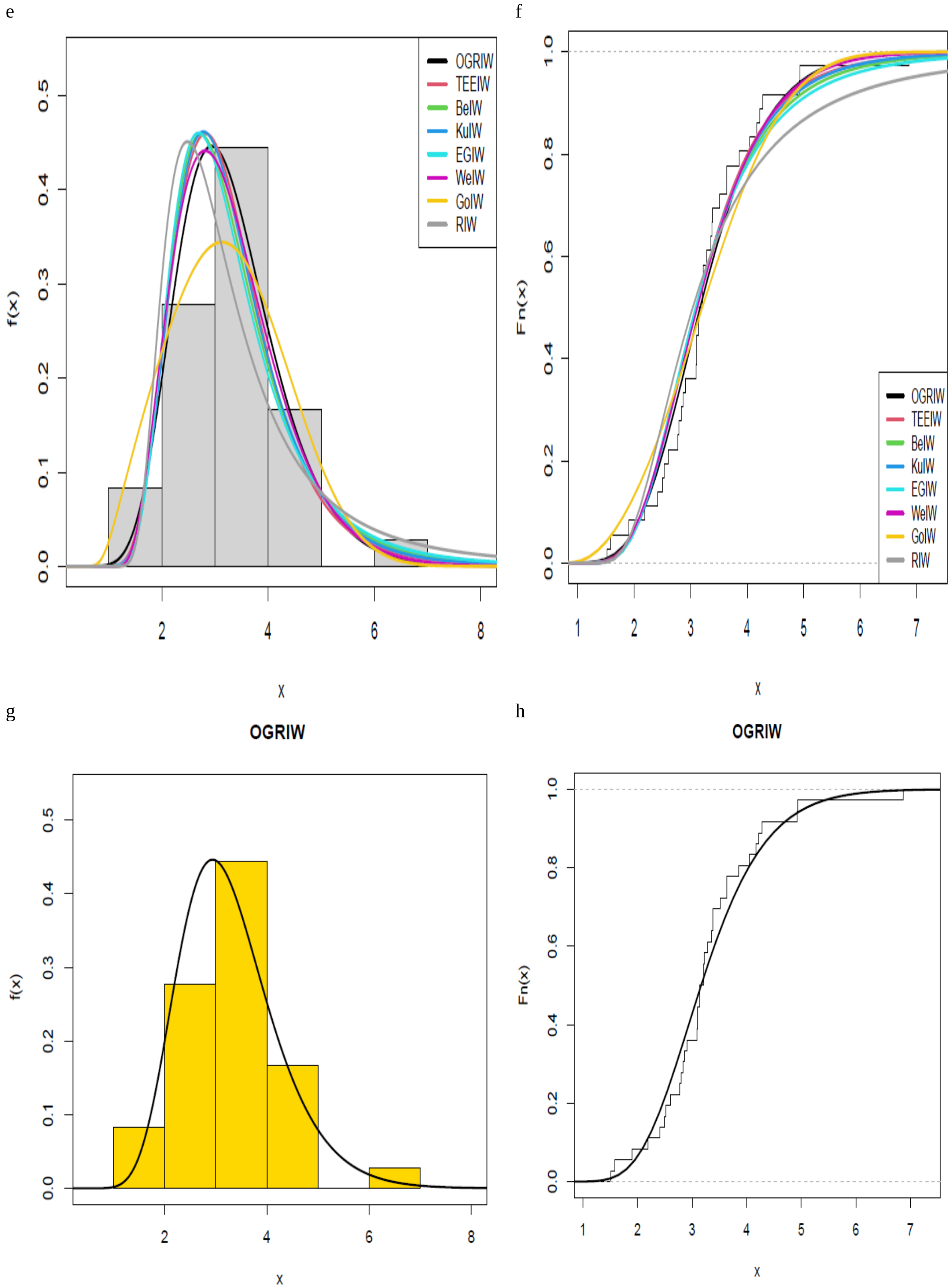

FIGURE 3.- (e) and (g) Estimated PDF and (f) and (h) CDF for data 2.

6. Conclusions

This paper introduces a novel family of continuous distributions called the Odd Generalized Rayleigh-G Family, with a special sub-model known as the odd Generalized Rayleigh Inverse Weibull (OGRIW) distribution. The OGRIW distribution is characterised by various mathematical properties, including moments, moment generating function, incomplete moments, quantile function, order statistics and Rényi entropy. The parameters of the OGRIW distribution are estimated using the Maximum Likelihood Estimation method.

The application of the proposed OGRIW model to COVID-19 mortality rate data from Mexico and Canada demonstrates its superiority over several competitive models. When compared to alternatives, including the Truncated Exponentiated Exponential Inverse Weibull distribution, Beta Inverse Weibull distribution, Kumaraswamy Inverse Weibull distribution, Exponential Generalized Inverse Weibull distribution, Weibull Inverse Weibull distribution, Rayleigh Inverse Weibull distribution, Gompertz Inverse Weibull distribution, Burr type X distribution, Rayleigh distribution and Inverse Weibull distribution, the findings suggest that the OGRIW model provides a better fit and more accurate representation of the COVID-19 mortality rate data.

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CONFLICTS OF INTEREST

None

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