

## Picard and Adomian decomposition methods for a fractional quadratic integral equation via $\zeta$ -generalized $\xi$ -fractional integral.

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**ABSTRACT:** The primary focus of this paper is to thoroughly examine and analyze a class of a fractional quadratic integral equation via  $\zeta$ -generalized  $\xi$ -fractional integral. To achieve this, we introduce an operator that possesses fixed points corresponding to the solutions of the fractional quadratic integral equation, effectively transforming the given equation into an equivalent fixed-point problem. By applying the Banach fixed-point theorems, we prove the uniqueness of solutions to fractional quadratic integral equation. Additionally, The Adomian decomposition method is used, to solve the resulting fractional quadratic integral equation. This technique rapidly provides convergent successive approximations of the exact solution to the given fractional quadratic integral equation, therefore, we investigate the convergence of approximate solutions, using the Adomian decomposition method. Finally, we provide some examples, to demonstrate our results. Our findings contribute to the current understanding of fractional quadratic integral equation and their solutions and have the potential to inform future research in this area.

**Keywords:**  $\xi$ - fractional operator; Fractional differential equations; Monotone operator; fixed point theorems

### 1. INTRODUCTION

The emergence of various novel definitions for fractional integrals and derivatives opens up opportunities for further exploration and research, with the possibility of developing new ideas and techniques utilizing these new fractional operators. In recent decades, there has been a significant surge in interest in the Adomian decomposition method (ADM) as a powerful tool for solving various types of functional equations. This approach has been successfully applied in numerous studies in the field of applied sciences. Interested readers can refer to the works by [1–4] for further details on this subject. Additionally, quadratic integral equations (QIEs) have proven to be highly useful in various applications, including the theory of radiative exchange, dynamic theory of gases, and traffic theory, among others. Several monographs and articles have been dedicated to the study of QIEs, and interested readers can find more information in references such as [5–8]. Picard's Method (PM) is another technique highlighted in [9], which involves using a series of increasingly specific algebraic approximations to obtain an accurate solution to a differential equation with a first-order initial value.

The field of Fractional Calculus (FC) has a rich history and has gained recognition in multiple scientific domains, such as mathematics, physics, and engineering. Over time, new fractional integrals and derivatives have emerged, showcasing the vast range of definitions and their numerous practical applications. Additional information on this topic can be found in a series of books and research papers, namely [10–14]. Conversely, there are various methods available to create a

generalization of classical fractional integrals. One common approach is to introduce parameters into classical definitions or a specific function, as mentioned in [15].

Furthermore, in their work [16], the authors presented a generalized version of fractional integrals on a specific space by introducing a parameter. They termed these new definitions as generalized  $\xi$ -fractional integrals.

For instance, the investigators in [7] found the uniqueness of the solution for

$$u(s) = \varphi(s) + v(s, u(s)) \int_0^s \varpi(\kappa, u(\kappa)) d\kappa,$$

by applying the Picard method and Adomian method. In [22], the researchers examined the approximate and analytical solutions for the fractional quadratic integral equation (FQIE)

$$u(s) = \varphi(s) + v(s, u(s)) \mathcal{I}_{0+}^{\sigma; \vartheta} \varpi(s, u(s)), \quad s \in \mathbf{J} = [0, 1], \quad \sigma > 0,$$

where  $\mathcal{I}_{0+}^{\sigma; \vartheta}$  is the Katugampola fractional integral. In [23], Rashid et al. introduce another so-called  $\zeta$ -Generalized  $\xi$ -fractional Integral and gave some important properties concerning this type of fractional operator. We direct the readers to the papers [18, 19] and the references therein for further results based on this operator. Also, in [20], Abood et.al, given the analytical and approximate solutions for the following fractional quadratic integral equation

$$u(s) = \varphi(s) + v(s, u(s)) \mathcal{I}^{\sigma; \xi} \varpi(s, u(s)), \quad s \in \mathbf{J} = [0, 1], \quad \sigma, \xi > 0,$$

where  $\mathcal{I}^{\sigma; \xi}$  is the left sided  $\xi$ -RL fractional integral of order  $\sigma$ .

Inspired by the above works, we investigate in this paper the exact and approximate solutions for the FQIE

$$u(s) = \varphi(s) + v(s, u(s)) \mathcal{I}^{\sigma; \zeta; \xi} \varpi(s, u(s)), \quad s \in \mathbf{J} = [0, 1], \quad \sigma, \zeta > 0, \tag{1}$$

where  $\mathcal{I}^{\sigma; \zeta; \xi}$  is The  $\zeta$ -generalized  $\xi$ - fractional integral operator of a function  $\varpi$  of order  $\sigma$ , as defined in Definition (1).

The remainder of the paper is structured as follows. In Section 2, we review some fractional calculus notations, definitions, and lemmas that are relevant to our research. Section 3, contains the primary uniqueness results for Eq. (1), attained by applying the Banach fixed-point theorems. This section also contains an analysis of the methods of Adomian decomposition and Picard for fractional quadratic integral equation via  $\zeta$ -generalized  $\xi$ -fractional integral. Section 4, includes some examples to demonstrate and elucidate the findings. Concluding remarks are presented in Section 5.

## 2. PRELIMINARIES:

To begin, we establish the notation, definitions, and fundamental concepts that will be utilized throughout this paper.

Consider the space  $X_{\xi}^P([a, b], \mathbb{R})$ , ( $c \in \mathbb{R}, 1 \leq P \leq \infty$ ) of those real-valued Lebesgue measurable functions  $g$  on  $[a, b]$  for which  $\|g\|_{X_{\xi}^P} < \infty$ , where the norm is defined by

$$\|g\|_{X_{\xi}^P} = \left( \int_a^b \xi'(s) |g(s)|^P ds \right)^{\frac{1}{P}},$$

where  $\xi$  is an increasing and positive function on  $[a, b]$  such that  $\xi'$  is continuous on  $[a, b]$  with  $\xi(0) = 0$ . In particular, when  $\xi(s) = s$ , the space  $X_{\xi}^P(a, b)$  coincides with the  $L_P(a, b)$  space.

For  $\varsigma = \frac{\beta(\zeta - \sigma) + \sigma}{\zeta}$ ,  $0 < \sigma < 1$ , and  $0 \leq \beta \leq 1$ . Let  $\xi \in C^1(\mathbf{J}, \mathbb{R})$  be an increasing function with  $\xi'(s) \neq 0$ , for all  $s \in \mathbf{J}$ , we give the definition of the fractional operators used throughout this paper.

**Definition 1.** [23] ( $\zeta$ -Generalized  $\xi$ -fractional Integral). Let  $\varpi \in X_{\xi}^P(\mathbf{J}, \mathbb{R})$  and  $\mathbf{J}$  be a finite or infinite interval on the real axis  $\mathbb{R}$ ,  $\xi(s) > 0$  be an increasing function on  $(a, b)$ , and  $\xi'(\kappa) > 0$  be continuous on  $(a, b)$  and  $\sigma > 0$ . The generalized  $\zeta$ -fractional integral operator of a function  $\varpi$  of order  $\sigma$  given by

$$\mathcal{I}^{\sigma; \zeta; \xi} \varpi(s, u(s)) = \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_a^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} \varpi(\kappa, u(\kappa)) d\kappa, \quad \zeta > 0, \tag{2}$$

where  $\Gamma_{\zeta}(\cdot)$  denotes the  $\zeta$ -Gamma function [17] define by

$$\Gamma_{\zeta}(\sigma) = \int_0^{\infty} \kappa^{\sigma-1} e^{-\frac{\kappa}{\zeta}} d\kappa, \quad \text{Re}(\sigma) > 0.$$

**Lemma 1.** [24, 25] Let  $\sigma, \eta, \delta > 0$ , and  $\frac{\delta}{\zeta} > -1$ , Then,

1.  $\mathcal{I}^{\sigma; \zeta; \xi} \mathcal{I}^{\eta; \zeta; \xi} \varpi(s, u(s)) = \mathcal{I}^{\sigma+\eta; \zeta; \xi} \varpi(s, u(s)) = \mathcal{I}^{\eta; \zeta; \xi} \mathcal{I}^{\sigma; \zeta; \xi} \varpi(s, u(s)).$
2.  $\mathcal{I}^{\sigma; \zeta; \xi} (\xi(s) - \xi(a))^{\frac{\delta}{\zeta}} = \frac{\Gamma_{\zeta}(\delta + \zeta)}{\Gamma_{\zeta}(\sigma + \delta + \zeta)} (\xi(s) - \xi(a))^{\frac{\sigma + \delta}{\zeta}}.$

### 3. MAIN RESULTS

For fulfillment the main results, we shall pose the following hypotheses:

(H<sub>1</sub>) The function  $\varphi : J \rightarrow \mathbb{R}$  is continuous on  $J$ ,

(H<sub>2</sub>) The functions  $\varpi, v: J \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous and bounded with

$$\ell_1 = \sup_{(s,u) \in J \times \mathbb{R}} |v(s,u)|, \text{ and } \ell_2 = \sup_{(s,u) \in J \times \mathbb{R}} |\varpi(s,u)|.$$

(H<sub>3</sub>) There exists two constants  $\mu_1, \mu_2 > 0$  such that

$$\begin{aligned} |v(s,u) - v(s,y)| &\leq \mu_1 |u - y|, \\ |\varpi(s,u) - \varpi(s,y)| &\leq \mu_2 |u - y|, \end{aligned}$$

for any  $s \in J$  and  $u, y \in \mathbb{R}$ .

**Lemma 2.** Suppose that assumption (H<sub>2</sub>) hold. The function  $u(s)$  is solution of Eq. (1) if and only if  $u(s)$  is a fixed-point of the operator  $F : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$  defined by

$$(Fu)(s) = \varphi(s) + v(s, u(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u(\kappa)) d\kappa, \quad s \in J, \quad \sigma, \zeta > 0.$$

#### 3.1 EXISTENCE AND UNIQUENESS OF SOLUTION FOR EQ.(1):

Based on Banach’s fixed point theorem [21], we are going to prove the unique solution of Eq. (1).

**Theorem 1.** (Uniqueness Theorem) Assume the hypotheses (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>3</sub>) hold. If

$$\Pi := [\mu_1 \ell_2 + \mu_2 \ell_1] \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} < 1, \tag{3}$$

then the FQIE (1) has a unique solution  $u \in C(J)$ .

*Proof.* It is easy to see that  $F : C(J) \rightarrow C(J)$ . Now, let  $\mathfrak{B}_\varepsilon \subset C(J)$  where  $\mathfrak{B}_\varepsilon$  define as

$$\mathfrak{B}_\varepsilon = \{u(s) \in C(J) : |u(s) - \varphi(s)| \leq \varepsilon, \text{ for } s \in J\}.$$

If we choose  $\varepsilon = \frac{\ell_1 \ell_2 \Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)}$ . Then, the operator  $F : \mathfrak{B}_\varepsilon \rightarrow \mathfrak{B}_\varepsilon$ . Indeed, for  $u \in \mathfrak{B}_\varepsilon$ , and from Lemma (1-2), we have

$$\begin{aligned} |u(s) - \varphi(s)| &\leq |v(s, u(s))| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |\varpi(\kappa, u(\kappa))| d\kappa \\ &\leq \ell_1 \ell_2 \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} d\kappa \\ &\leq \frac{\ell_1 \ell_2 \Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} (\xi(s) - \xi(0))^{\frac{\sigma}{\zeta}} \\ &\leq \frac{\ell_1 \ell_2 \Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} = \varepsilon. \end{aligned}$$

Besides,  $\mathfrak{B}_\varepsilon \subseteq C(\mathcal{J})$ . For proving  $F$  is a contraction, we have

$$\begin{aligned} (Fu)(s) - (Fy)(s) &= v(s, u(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u(\kappa)) d\kappa \\ &\quad - v(s, y(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, y(\kappa)) d\kappa \\ &\quad + v(s, u(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, y(\kappa)) d\kappa \\ &\quad - v(s, u(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, y(\kappa)) d\kappa \\ &= [v(s, u(s)) - v(s, y(s))] \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, y(\kappa)) d\kappa \\ &\quad + v(s, u(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} [\varpi(\kappa, u(\kappa)) - \varpi(\kappa, y(\kappa))] d\kappa. \end{aligned}$$

Then,

$$\begin{aligned} |(Fu)(s) - (Fy)(s)| &\leq |v(s, u(s)) - v(s, y(s))| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |\varpi(\kappa, y(\kappa))| d\kappa \\ &\quad + |v(s, u(s))| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |\varpi(\kappa, u(\kappa)) - \varpi(\kappa, y(\kappa))| d\kappa \\ &\leq \mu_1 \ell_2 |u(s) - y(s)| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} d\kappa \\ &\quad + \mu_2 \ell_1 |u(\kappa) - y(\kappa)| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} d\kappa \\ &\leq [\mu_1 \ell_2 + \mu_2 \ell_1] |u(s) - y(s)| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} d\kappa, \end{aligned}$$

which implies

$$\begin{aligned} \|(Fu)(s) - (Fy)(s)\| &= \sup_{s \in \mathcal{J}} |(Fu)(s) - (Fy)(s)| \\ &\leq [\mu_1 \ell_2 + \mu_2 \ell_1] \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \|u - y\| \\ &= \Pi \|u - y\|. \end{aligned}$$

By (3),  $F$  is a contraction mapping. Hence, by Banach's contraction principle,  $F$  has a unique fixed point  $u \in C(\mathcal{J})$ , which is a solution to the FQIE (1).

### 3.2 PICARD METHOD (PM):

By using the PM to Eq. (1), the solution is given by the sequence.

$$\begin{aligned} u_n(s) &= \varphi(s) + v(s, u_{n-1}(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-1}(\kappa)) d\kappa, \quad n = 1, 2, \dots \\ u_0(s) &= \varphi(s) \end{aligned} \tag{4}$$

The functions  $u_n$  can be written as

$$u_n = u_0 + \sum_{j=1}^n [u_j - u_{j-1}],$$

where the functions  $\{u_n(s)\}_{n \geq 1}$  are continuous.

Here, the sequence functions  $u_n(s)$  will converge to  $u(s)$ , If the infinite series  $\sum_{j=1}^{\infty} [u_j - u_{j-1}]$  converges. Thus, the solution will be given by

$$u(s) = \lim_{n \rightarrow \infty} u_n(s).$$

Now, we prove that  $\{u_n(s)\}_{n \geq 1}$  is uniform convergence. Consider the infinite series

$$\sum_{n=1}^{\infty} [u_n(s) - u_{n-1}(s)].$$

From (4) for  $n = 1$ , we achieve

$$u_1(s) - u_0(s) = v(s, u_0(s)) \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} \varpi(\kappa, u_0(\kappa)) d\kappa.$$

Consequently,

$$|u_1(s) - u_0(s)| \leq \ell_1 \ell_2 \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} d\kappa \leq \ell_1 \ell_2 \frac{\Gamma_{\zeta}(\zeta)}{\Gamma_{\zeta}(\sigma + \zeta)} (\xi(s))^{\frac{\sigma}{\zeta}}. \tag{5}$$

Here, we find the expression  $u_n(s) - u_{n-1}(s)$ , for  $n \geq 2$  as

$$\begin{aligned} u_n(s) - u_{n-1}(s) &= v(s, u_{n-1}(s)) \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-1}(\kappa)) d\kappa \\ &\quad - v(s, u_{n-2}(s)) \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-2}(\kappa)) d\kappa \\ &\quad + v(s, u_{n-1}(s)) \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-2}(\kappa)) d\kappa \\ &\quad - v(s, u_{n-1}(s)) \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-2}(\kappa)) d\kappa \\ &= v(s, u_{n-1}(s)) \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} [\varpi(\kappa, u_{n-1}(\kappa)) - \varpi(\kappa, u_{n-2}(\kappa))] d\kappa \\ &\quad + [v(s, u_{n-1}(s)) - v(s, u_{n-2}(s))] \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-2}(\kappa)) d\kappa. \end{aligned}$$

Using hypotheses (2) and (3), we attain

$$\begin{aligned} |u_n(s) - u_{n-1}(s)| &\leq |v(s, u_{n-1}(s))| \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} |\varpi(\kappa, u_{n-1}(\kappa)) - \varpi(\kappa, u_{n-2}(\kappa))| d\kappa \\ &\quad + |v(s, u_{n-1}(s)) - v(s, u_{n-2}(s))| \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} |\varpi(\kappa, u_{n-2}(\kappa))| d\kappa \\ &\leq \mu_2 \ell_1 \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} |u_{n-1}(\kappa) - u_{n-2}(\kappa)| d\kappa \\ &\quad + \mu_1 \ell_2 |u_{n-1}(s) - u_{n-2}(s)| \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1 - \frac{\sigma}{\zeta}}} d\kappa. \end{aligned}$$

By taking  $n = 2$ , then, using (5), we get

$$\begin{aligned}
 |u_2(s) - u_1(s)| &\leq \mu_2 \ell_1 \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |u_1(\kappa) - u_0(\kappa)| d\kappa \\
 &\quad + \mu_1 \ell_2 |u_1(s) - u_0(s)| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} d\kappa \\
 &\leq \mu_2 \ell_1^2 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} (\xi(\kappa))^{\frac{\sigma}{\zeta}} d\kappa \\
 &\quad + \mu_1 \ell_2^2 \ell_1 \left( \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} (\xi(s))^{\frac{\sigma}{\zeta}} \right) \left( \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} (\xi(s))^{\frac{\sigma}{\zeta}} \right) \\
 &\leq \mu_2 \ell_1^2 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \frac{\Gamma_\zeta(\sigma + \zeta)}{\Gamma_\zeta(2\sigma + \zeta)} (\xi(s))^{2\frac{\sigma}{\zeta}} \\
 &\quad + \mu_1 \ell_2^2 \ell_1 \frac{\Gamma_\zeta(\zeta) \Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta) \Gamma_\zeta(\sigma + \zeta)} (\xi(s))^{2\frac{\sigma}{\zeta}} \\
 &\leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \left[ \mu_2 \ell_1 \frac{\Gamma_\zeta(\sigma + \zeta)}{\Gamma(2\sigma + \zeta)} + \mu_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma(\sigma + \zeta)} \right] (\xi(s))^{2\frac{\sigma}{\zeta}}.
 \end{aligned}$$

Similarly, for  $n = 3$

$$\begin{aligned}
 |u_3(s) - u_2(s)| &\leq \mu_2 \ell_1 \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |u_2(s) - u_1(s)| d\kappa \\
 &\quad + \mu_1 \ell_2 |u_2(s) - u_1(s)| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} d\kappa \\
 &\leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \left( \mu_2 \ell_1 \frac{\Gamma_\zeta(\sigma + \zeta)}{\Gamma_\zeta(2\sigma + \zeta)} + \mu_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \right) \\
 &\quad \times \left( \mu_2 \ell_1 \frac{\Gamma_\zeta(2\sigma + \zeta)}{\Gamma_\zeta(3\sigma + \zeta)} + \frac{\mu_1 \ell_2 \Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \right) (\xi(s))^{3\frac{\sigma}{\zeta}}.
 \end{aligned}$$

Repeating the same process, we get

$$\begin{aligned}
 |u_n(s) - u_{n-1}(s)| &\leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \left( \mu_2 \ell_1 \frac{\Gamma_\zeta(\sigma + \zeta)}{\Gamma_\zeta(2\sigma + \zeta)} + \mu_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \right) \\
 &\quad \times \left( \mu_2 \ell_1 \frac{\Gamma_\zeta(2\sigma + \zeta)}{\Gamma_\zeta(3\sigma + \zeta)} + \mu_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \right) \times \dots \\
 &\quad \times \left( \mu_2 \ell_1 \frac{\Gamma_\zeta((n-1)\sigma + \zeta)}{\Gamma_\zeta(n\sigma + \zeta)} + \mu_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \right) (\xi(s))^{n\frac{\sigma}{\zeta}} \\
 &\leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \left( \mu_2 \ell_1 \frac{\Gamma_\zeta(\sigma + \zeta)}{\Gamma_\zeta(\sigma + \zeta)} + \mu_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \right) \\
 &\quad \times \left( \mu_2 \ell_1 \frac{\Gamma_\zeta(2\sigma + \zeta)}{\Gamma_\zeta(2\sigma + \zeta)} + \mu_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \right) \times \dots \\
 &\quad \times \left( \mu_2 \ell_1 \frac{\Gamma_\zeta((n-1)\sigma + 1)}{\Gamma_\zeta((n-1)\sigma + 1)} + \frac{\mu_1 \ell_2}{\Gamma_\zeta(\sigma + 1)} \right) \\
 &\leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} ((\mu_2 \ell_1 + \mu_1 \ell_2)) \times ((\mu_2 \ell_1 + \mu_1 \ell_2)) \times \dots \\
 &\quad \times ((\mu_2 \ell_1 + \mu_1 \ell_2)) \\
 &\leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} ((\mu_2 \ell_1 + \mu_1 \ell_2))^n.
 \end{aligned}$$

Since  $[\mu_1 \ell_2 + \mu_2 \ell_1] \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} < 1$  (3), then the series  $\sum_{n=1}^\infty [u_n(s) - u_{n-1}(s)]$  and the sequence  $\{u_n(s)\}$  are uniformly convergent.

Because  $\varpi(s, u)$  and  $v(s, u)$  are continuous in  $u$ , it follows that

$$\begin{aligned} u(s) &= \lim_{n \rightarrow \infty} v(s, u_n(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u_n(\kappa)) d\kappa \\ &= v(s, u(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u(\kappa)) d\kappa. \end{aligned}$$

This demonstrates the existence of a solution.

Here, we will prove the solution  $u(s)$  is unique, so, we will take  $y(s)$  which is a continuous solution to Eq. (1) and define it as

$$y(s) = \varphi(s) + v(s, y(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, y(\kappa)) d\kappa, \quad s \in [0, 1], \sigma > 0.$$

Hence,

$$\begin{aligned} y(s) - u_n(s) &= v(s, y(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, y(\kappa)) d\kappa \\ &\quad - v(s, u_{n-1}(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-1}(\kappa)) d\kappa \\ &\quad + v(s, y(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-1}(\kappa)) d\kappa \\ &\quad - v(s, y(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-1}(\kappa)) d\kappa \\ &= v(s, y(s)) \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} [\varpi(\kappa, y(\kappa)) - \varpi(\kappa, u_{n-1}(\kappa))] d\kappa \\ &\quad + [v(s, y(s)) - v(s, u_{n-1}(s))] \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} \varpi(\kappa, u_{n-1}(\kappa)) d\kappa. \end{aligned}$$

Applying hypotheses (H<sub>2</sub>) and (H<sub>3</sub>), we get

$$\begin{aligned} |y(s) - u_n(s)| &\leq |v(s, y(s))| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |\varpi(\kappa, y(\kappa)) - \varpi(\kappa, u_{n-1}(\kappa))| d\kappa \\ &\quad + |v(s, y(s)) - v(s, u_{n-1}(s))| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |\varpi(\kappa, u_{n-1}(\kappa))| d\kappa \\ &\leq \mu_2 \ell_1 \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} |y(\kappa) - u_{n-1}(\kappa)| d\kappa \\ &\quad + \mu_1 \ell_2 |y(s) - u_{n-1}(s)| \frac{1}{\zeta \Gamma_\zeta(\sigma)} \int_0^s \frac{\xi'(\kappa)}{(\xi(s) - \xi(\kappa))^{1-\frac{\sigma}{\zeta}}} d\kappa. \end{aligned} \tag{6}$$

But we have

$$|y(s) - \varphi(s)| \leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} \xi(s)^{\frac{\sigma}{\zeta}}.$$

Hence, with using (6), we get

$$|y(s) - u_n(s)| \leq \ell_1 \ell_2 \frac{\Gamma_\zeta(\zeta)}{\Gamma_\zeta(\sigma + \zeta)} [(\mu_2 \ell_1 + \mu_1 \ell_2)]^n.$$

Consequently,

$$\lim_{n \rightarrow \infty} u_n(s) = y(s) = u(s).$$

**Corollary 1.** According to the hypotheses of the Theorem 1. If  $\xi(s) = s^\theta$  and  $\zeta = 1$ , then Eq. (1) reduces to

$$u(s) = \varphi(s) + v(s, u(s)) \int_0^s \frac{\kappa^{\theta-1}}{\Gamma(\sigma)} \left( \frac{s^\theta - \kappa^\theta}{\theta} \right)^{\sigma-1} \varpi(\kappa, u(\kappa)) d\kappa,$$

which has a unique solution see [22].

### 3.3 ADOMIAN DECOMPOSITION METHOD:

The Adomian decomposition method (ADM) is a mathematical technique used for solving nonlinear fractional differential equations. One of the notable advantages of the ADM is its simplicity and ease of application. Here are some key aspects that are often considered positive:

**Analytical Solution:** The advantage of ADM is that it provides an analytical rather than purely numerical solution, which may provide insights into the system’s behavior and make it more interpretable than a purely numerical solution.

**Nonlinear Problem Solving:** In many fields of science and engineering where nonlinearity is prevalent, ADM is particularly useful for solving nonlinear differential equations. Also, there are a lot of methods which valid with deal solve fractional differential equations.

In this subsection, we shall analyze ADM for Eq. (1). The solution of the FQEI (1) by using ADM given by

$$u_0(s) = \varphi(s), \tag{7}$$

$$u_{\Theta}(s) = \chi_{(\Theta-1)}(s) \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \kappa_{(\Theta-1)}(s), \tag{8}$$

where  $\chi_{\Theta}$  and  $\kappa_{\Theta}$  are Adomian polynomials of the nonlinear terms  $v(s, u)$  and  $\varpi(\kappa, u)$ , which are given as follows

$$\chi_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left( v \left( s, \sum_{\Theta=0}^{\infty} \lambda^{\Theta} u_{\Theta} \right) \right) \right]_{\lambda=0}, \tag{9}$$

$$\kappa_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left( \varpi \left( s, \sum_{\Theta=0}^{\infty} \lambda^{\Theta} u_{\Theta} \right) \right) \right]_{\lambda=0}. \tag{10}$$

Following, we will show the solution as

$$u(s) = \sum_{\Theta=0}^{\infty} u_{\Theta}. \tag{11}$$

### 3.4 CONVERGENCE ANALYSIS:

Convergence analysis helps in understanding how sensitive a numerical solution is to changes in input parameters. This information is crucial for identifying the stability and reliability of a numerical method under various conditions, so, here we will study the converges of the solution (11) of Eq. (1) by using ADM.

**Theorem 2.** Let  $u(s)$  be a solution of Eq. (1) and there exists a constant  $\mathcal{A} > 0$  satisfy  $|u_1(s)| < \mathcal{A}$ . Then, the solution (11) of Eq. (1) using ADM converges.

*Proof.* Put  $\{n_{q_1}\}$  be a sequence such that  $n_{q_1} = \sum_{\Theta=0}^{q_1} u_{\Theta}$  is a sequence of partial sums from the series (11) and we have

$$v(s, u) = \sum_{\Theta=0}^{\infty} \chi_{\Theta},$$

$$\varpi(s, u) = \sum_{\Theta=0}^{\infty} \kappa_{\Theta}.$$



Let  $n_{q_1}$  and  $n_{q_2}$  be two partial sums with  $q_1 > q_2$ . Now, we prove that  $n_{q_1}$  is a Cauchy sequence in  $C(J)$ .

$$\begin{aligned}
 n_{q_1} - n_{q_2} &= \sum_{\Theta=0}^{q_1} u_{\Theta} - \sum_{\Theta=0}^{q_2} u_{\Theta} \\
 &= \sum_{\Theta=0}^{q_1} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) \right) - \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_2} \varkappa_{(\Theta-1)}(s) \right) \\
 &= \sum_{\Theta=0}^{q_1} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) \right) - \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) \right) \\
 &\quad + \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) \right) - \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_2} \varkappa_{(\Theta-1)}(s) \right) \\
 &= \left[ \sum_{\Theta=0}^{q_1} \chi_{(\Theta-1)}(s) - \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \right] \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) \right) \\
 &\quad + \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \left[ \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) - \sum_{\Theta=0}^{q_2} \varkappa_{(\Theta-1)}(s) \right] \right)
 \end{aligned}$$

But,

$$\begin{aligned}
 \|n_{q_1} - n_{q_2}\| &\leq \max_{s \in J} \left| \sum_{\Theta=q_2+1}^{q_1} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) \right) \right| \\
 &\quad + \max_{s \in J} \left| \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \left( \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=q_2+1}^{q_1} \varkappa_{(\Theta-1)}(s) \right) \right| \\
 &\leq \max_{s \in J} \left| \sum_{\Theta=q_2}^{q_1-1} \chi_{\Theta}(s) \right| \left| \mathcal{G}_{0^+}^{\sigma, \zeta; \xi} \sum_{\Theta=0}^{q_1} \varkappa_{(\Theta-1)}(s) \right| + \max_{s \in J} \left| \sum_{\Theta=0}^{q_2} \chi_{(\Theta-1)}(s) \right| \left| \sum_{\Theta=q_2}^{q_1-1} \varkappa_{\Theta}(s) \right| \\
 &\leq \max_{s \in J} |v(s, n_{q_1-1}) - v(s, n_{q_2-1})| \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(k)}{(\xi(s) - \xi(k))^{1-\frac{\sigma}{\zeta}}} |\varpi(k, n_{q_1})| dk \\
 &\quad + \max_{s \in J} |v(s, n_{q_2})| \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(k)}{(\xi(s) - \xi(k))^{1-\frac{\sigma}{\zeta}}} |\varpi(k, n_{q_1-1}) - \varpi(k, n_{q_2-1})| dk \\
 &\leq \mu_1 \ell_2 \max_{s \in J} |n_{q_1-1} - n_{q_2-1}| \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(k)}{(\xi(s) - \xi(k))^{1-\frac{\sigma}{\zeta}}} dk \\
 &\quad + \mu_2 \ell_1 \max_{s \in J} |n_{q_1-1} - n_{q_2-1}| \frac{1}{\zeta \Gamma_{\zeta}(\sigma)} \int_0^s \frac{\xi'(k)}{(\xi(s) - \xi(k))^{1-\frac{\sigma}{\zeta}}} dk \\
 &\leq \frac{\Gamma_{\zeta}(\zeta)}{\Gamma_{\zeta}(\sigma + \zeta)} [(\mu_2 \ell_1 + \mu_1 \ell_2)] \max_{s \in J} |n_{q_1-1} - n_{q_2-1}| \\
 &\leq \Pi \|n_{q_1-1} - n_{q_2-1}\|.
 \end{aligned}$$

Let  $q_1 = q_2 + 1$  then

$$\|n_{q_2+1} - n_{q_2}\| \leq \Pi \|n_{q_2} - n_{q_2-1}\| \leq \Pi^2 \|n_{q_2-1} - n_{q_2-2}\| \leq \dots \leq \Pi^{q_2} \|n_1 - n_0\|.$$

Also, we have

$$\begin{aligned}
 \|n_{q_1} - n_{q_2}\| &\leq \|n_{q_2+1} - n_{q_2}\| + \|n_{q_2+2} - n_{q_2+1}\| + \dots + \|n_{q_1} - n_{q_1-1}\| \\
 &\leq [\Pi^{q_2} + \Pi^{q_2+1} + \dots + \Pi^{q_1-1}] \|n_1 - n_0\| \\
 &\leq \Pi^{q_2} [1 + \Pi + \dots + \Pi^{q_1-q_2-1}] \|n_1 - n_0\| \\
 &\leq \Pi^{q_2} \left[ \frac{1 - \Pi^{q_1-q_2}}{1 - \Pi} \right] \|u_1\|.
 \end{aligned}$$

The assumptions  $0 < \Pi < 1$ , and  $q_1 > q_2$  lead to  $(1 - \Pi^{q_1 - q_2}) \leq 1$ . Hence,

$$\begin{aligned} \|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| &\leq \frac{\Pi^{q_2}}{1 - \Pi} \|u_1\| \\ &\leq \frac{\Pi^{q_2}}{1 - \Pi} \max_{s \in J} |u_1(s)|. \end{aligned}$$

But  $|u_1(s)| < \mathcal{A}$  and as  $q_2 \rightarrow \infty$ , then,  $\|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\| \rightarrow 0$  and hence,  $\{\mathcal{N}_{q_1}\}$  is a Cauchy sequence in  $C(J)$  and the series  $\sum_{\Theta=0}^{\infty} u_{\Theta}(s)$  converges.

#### 4. EXAMPLES:

In this section, we present some examples to support our obtained results in the previous section.

**Example 1.** Consider the following nonlinear FQIE,

$$u(s) = \left( s^3 - \frac{104s^{\frac{17}{2}}}{750} \right) + \frac{1}{4} u(s) \mathcal{G}_{0^+}^{\frac{1}{2}, 2; \frac{1}{2}} u^4(s). \tag{12}$$

Here, the  $u(s) = s^3$  is the exact solution for (12).

Take  $\xi = \frac{1}{2}$  and  $\zeta = 2$ , and applying PM to (12), we get

$$\begin{aligned} u_n(s) &= \left( s^2 - \frac{208s^{\frac{23}{3}}}{1500} \right) + \frac{1}{6} u_{n-1}(s) \mathcal{G}_{0^+}^{\frac{1}{2}, 2; \frac{1}{2}} u_{n-1}^4(s), \quad n = 1, 2, \dots, \\ u_0(s) &= \left( s^2 - \frac{208s^{\frac{23}{3}}}{1500} \right). \end{aligned}$$

and the solution will be in the form

$$u(s) = u_n(s).$$

Using ADM on (12), we get

$$\begin{aligned} u_0(s) &= \left( s^2 - \frac{208s^{\frac{23}{3}}}{1500} \right), \\ u_i(s) &= \frac{1}{6} u_{i-1}(s) \mathcal{G}_{0^+}^{\frac{1}{2}, 2; \frac{1}{2}} \chi_{i-1}(s), \quad i = 1, 2, \dots, \end{aligned}$$

where  $\chi_i$  are Adomian polynomials of the nonlinear term  $u^4$ , and the solution will be

$$u(s) = \sum_{i=0}^p u_i(s).$$

**Example 2.** Consider the following nonlinear FQIE

$$u(s) = e^{s-2} + \sqrt[m]{u(s)} \mathcal{G}_{0^+}^{\frac{1}{2}, 2; \frac{1}{2}} (\tau + u^2(\tau)), \tag{13}$$

Applying Picard method to Eq. (13), we get

$$u_n(s) = e^{s-2} + \sqrt[m]{u_{n-1}(s)} \mathcal{G}_{0^+}^{\frac{1}{2}, 2; \frac{1}{2}} (\tau + u_{n-1}^2(\tau)),$$

with

$$u_0(s) = e^{s-2},$$

and the solution will be

$$u(s) = u_n(s).$$

Applying ADM to Eq. (13), we get

$$u_0(s) = e^{s-2},$$

and

$$u_j(s) = A_{j-1}(s) \mathcal{G}_{0^+}^{\frac{1}{2}, 2; \frac{1}{2}} (\tau + B_{j-1}(\tau)), \quad j \geq 1,$$

where  $A_j$  and  $B_j$  are Adomian polynomials of the nonlinear terms  $\sqrt[m]{u}$  and  $u^2$ , respectively, and the solution will be

$$u(s) = \sum_{i=0}^p u_i(s).$$

## 5. CONCLUSION:

In this article, we have addressed Eq. (1) that falls under the purview of on  $\zeta$ -generalized  $\xi$ -fractional integral. Firstly, we established the existence and uniqueness of solutions for Eq. (1) by developing and extending our approach by using the fixed point alternative. Next, we analyzed new specific outcomes that correspond to particular values of the parameters  $\xi$ ,  $\sigma$ , and  $\zeta$ . Our methodology employed the Banach fixed point theorem, Picard's method, and the Adomian decomposition method. Finally, we provided a practical examples that demonstrates the validity of our theoretical results. Our findings are novel for certain specific cases, which will facilitate the exploration of new applications and avenues for further research.

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The authors declare no conflict of interest.

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