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Homotopy analysis method for solving fuzzy Nonlinear Volterra integral equation of the second kind

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ABSTRACT: This paper introduces the homotopy analytic HAM, an effective and reliable method for solving computationally challenging fuzzy Volterra nonlinear integral equations of the second type. Numerical examples are provide to demonstrate the accuracy of HAM. To check for existence and uniqueness, the study employs the Banach fixed point theory and homotopy analysis. Finally the study resolves our issue using the MAPLE programme.

Keywords: -Fuzzy numbers, Fuzzy Volterra nonlinear and nonhomogeneous integral equations, Homotopy analysis method.

1. INTRODUCTION

Numerous significant problems in the disciplines under applied mathematics, including physics, medicine, and fuzzy control, greatly benefit from the study of fuzzy integral equations.. Von Zadeh [1] introduced the idea of fuzzy numbers and their calculation in 1965. And to day numerous fields use fuzzy sets. Although Kaleva [2] recommended fuzzy number operations and propose the fuzzy number concept, he opted to describe the integral of fuzzy functions using the Lebesgue-type integral notion. Recent years have seen a surge in research by mathematicians exploring numerical solutions for fuzzy integral equations and we employ HAM to resolve the nonlinear and nonhomogeneous fuzzy Volterra integral equation. This approach,which is used to solve Volterra integral equations of second kind, ,typically demonstrates avery high rate of convergence, with an unsteady hazy ,producing approximations with minimal errors in some iterations [3–4].

Integral equations are ubiquitous in mathematical modeling across science and engineering disciplines. Volterra integral equations of the second kind involve the unknown function under the integral and have broad applications in disciplines such as biology, mechanics, and finance [8]. These integro-differential equations can exhibit nonlinear behavior and do not always have analytical solutions, necessitating the use of numerical techniques [9]. Furthermore, ambiguity or vagueness in real-world systems may

require the incorporation of fuzzy parameters, increasing the complexity of obtaining accurate solutions [10].

The Thomotopy analysis method (HAM) has emerged as a versatile semi-analytical approach for handling nonlinear equations. By constructing a homotopy between the original equation and an initial approximation, HAM provides a framework for deriving convergence-controlled series solutions [11,12]. HAM has shown promising results for various types of fuzzy integral equations, including Fredholm and Volterra types [13,14]. However, the application of HAM for fuzzy nonlinear nonhomogeneous Volterra integral equations remains under explored. This paper aims to develop a numerical scheme based on HAM to solve such equations containing fuzzy parameters. The solution methodology involves formulating a homotopy equation, differentiating to obtain deformation equations, and iteratively calculating approximation terms. Convergence criteria and error analysis will be provided [15]. The proposed technique will be demonstrated through benchmark problems and compared to other existing methods like Adomian decomposition [16], finite differences [17], and fuzzy neural networks [18]. Challenges related to computational efficiency and optimal parameter selection will also be discussed [19]. Overall, this study provides a systematic framework for utilizing HAM to reliably solve fuzzy nonlinear nonhomogeneous Volterra integral equations arising in real-world contexts.

Challenges related to computational efficiency and selection of optimal parameters will also be discussed [20, 21]. Overall, this study will provide a systematic framework for utilizing HAM to reliably solve fuzzy nonlinear nonhomogeneous Volterra integral equations arising in real-world contexts[22].

2. MAIN CONCEPTS

The following introduces the fundamental definitions of a fuzzy number:

Definition 2.1. ([5]) $m: R \to [0,1]$ is a fuzzy number that meets the criteria below:

(1) m is upper semi-continuous function

(2) m(x)=0 outside some interval $[a, d]$.

(3) There are real numbers b and c such that $a \le b \le c \le d$, where

a- m(x) is a monotonically increasing function on [a,b]

b- m(x) is a monotonically decreasing function on $[c, d]$

c- For all $x \in [b, c]$, $m(x) = 1$.

E^1 represents the collection of all fuzzy numbers, which is a convex cone and is provided by Definition (2.1). Kaleva [2] offers an alternative definition of a fuzzy number's parameter.

Definition 2.2. A fuzzy number $\hat{m}$ in parametric form is a pair $(\underline{m}, \bar{m})$ of function $\underline{m}(\mu), \bar{m}(\mu), 0 \le \mu \le 1$, which satisfies the following:

i. $\underline{m}(\mu)$ is a non−decreasing, bounded, left−continuous function over $[0,1]$

ii. $\bar{m}(\mu)$ is a non− increasing, bounded left −continuous function over $[0,1]$

iii. $\underline{m}(\mu) \le \bar{m}(\mu), 0 \le \mu \le 1$

The term for the collection of all fuzzy numbers is denoted by E^1 [6].

Definition 2.3. For arbitrary fuzzy $m = (\underline{m}(\mu), \bar{m}(\mu)), n = (\underline{n}(\mu), \bar{n}(\mu)), 0 \le \mu \le 1$ and scalar w, subtraction, scalar product by w, addition and multiplication, respectively, are defined as follows:

1- Addition :

$$\underline{(m+n)}(\mu) = (\underline{m}(\mu) + \underline{n}(\mu)), \overline{(m+n)}(\mu) = (\bar{m}(\mu) + \bar{n}(\mu))$$

2- Subtraction :

$$\underline{(m-n)}(\mu) = (\underline{m}(\mu) - \underline{n}(\mu)), \overline{(m-n)}(\mu) = (\bar{m}(\mu) - \bar{n}(\mu))$$

3- Scalar product :

$$(w\hat{m})(\mu) = \begin{cases} (w\bar{m}(\mu), w\bar{m}(\mu)),\ (wm(\mu), w\underline{m}(\mu)), w \ge 0 \\ (w\underline{m}(\mu), w\bar{m}(\mu)),\ (w\bar{m}(\mu), w\underline{m}(\mu)), w < 0 \end{cases} \quad (1)$$

4-Multiplication:

$$(\tilde{m}.\tilde{n})(\mu) = \begin{cases} \underline{mn}(\mu) = \max\{\underline{m}(\mu)\underline{n}(\mu), \bar{m}(\mu)\underline{n}(\mu), \underline{m}(\mu)\bar{n}(\mu), \bar{m}(\mu)\bar{n}(\mu)\} \\ \overline{mn}(\mu) = \min\{\bar{m}(\mu)\bar{n}(\mu), \bar{m}(\mu)\underline{m}(\mu), \underline{m}(\mu)\bar{n}(\mu), \underline{m}(\mu)\underline{n}(\mu)\} \end{cases} \quad (2)$$

Definition 2.4. For arbitary Fuzzy numbers $\hat{m}, \hat{n} \in E^1$

$$D(\tilde{m}, \tilde{n}) = max\left\{ \sup_{\mu\in[0,1]} |\underline{m}(\mu) - \underline{n}(\mu)|, \sup_{\mu\in[0,1]} |\bar{m}(\mu) - \bar{n}(\mu)| \right\}, \quad (3)$$

The separation distance the $\hat{m}$ and $\hat{n}$, proves [7] that (E^1, D) are a complete metric spaces.

Definition 2.5. In ([16]), the Riemann integral is used to define the integral of a Fuzzy function $[a, b] \to E^1$. For a Fuzzy function, for each partition $p = \{x_0, \dots, x_n\}$ of $[a, b]$ and for arbitrary" $\xi_i \in [x_{i-1}, x_i], 1 \le i \le n$, let

$$R_p = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \quad (4)$$

$$\Delta := max\{|x_i - x_{i-1}|, 1 \le i \le n\}$$

$$\int_a^b f(x)\,dt = \lim_{\Delta\to 0} R_p\ ,\ h \to 0 \quad (5)$$

I the fuzzy function $f(x)$ it is continuous in metric D, then the integral certainly exists and in addition

$$\left(\int_a^b \underline{f(x;\mu)dt}\right) = \int_a^b \underline{f}(x;\mu)dx \text{ and } \left(\int_a^b \overline{f(x;\mu)}dx\right) = \int_a^b \bar{f}(x;\mu)dx \quad (6)$$

Where $\underline{f}(x;\mu), \bar{f}(x;\mu)$ is the form of parameter $f(x)$. It should be noted that the Lebesgue-type approach [6] can also be used to define the fuzzy integral. However, if $f(x)$ is continuous, both strategies result in same one. More information regarding characteristics of the fuzzy integral.

3. FUZZY NONLINEAR VOLTERRA INTEGRAL EQUATION OF THE SECOND-KIND

This paper covers the Complex nonlinear the Volterra integral equation of the second kind (FNVIE-2)—a fuzzy nonlinear equation with an integral kernel

$$\tilde{m}(x) = \tilde{f}(x) + \lambda\int_a^x k\left((x,t,\tilde{F}_1(x,t,\tilde{m}(t))) \cdot G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt \quad (7)$$

where $\tilde{m}(x)$ and $\tilde{f}(x)$ are fuzzy function on x : $a \le x \le b$, and where $\lambda \ge 0$, $k(x,t,\tilde{F}_1(\mu,t,\tilde{m}(t))).\tilde{G}\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)$ are analytic functions on $[a,b]\times[a,b]\times[a,b]\times E^n\times E^n$ and where $\tilde{F}_1(x,t,\tilde{m}(t)), \tilde{F}_2(t,s,\tilde{m}(s))$ are nonlinear function on $[a,b]$. For solving in the parameter form of Eq. (7), consider $(\underline{f}(x,\mu),\bar{f}(x,\mu))$ and $(\underline{m}(x,\mu),\bar{m}(x,\mu)), 0\le\mu\le 1$ and $t,s\in[a,b]$ are parameters of $\tilde{f}(x)$ and $\tilde{m}(x)$, respective. Then, the form of parameter of Eq. (7) is:

$$\underline{m}(x,\mu) = \underline{f}(x,\mu) + \lambda\int_a^x \underline{k\left((x,t,F_{1\alpha}(x,t,m(t,\mu))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,m(s,\mu))ds\right)dt}$$

$$\bar{m}(x,\mu) = \bar{f}(x,\mu) + \lambda\int_a^x \overline{k\left((x,t,F_{1\alpha}(x,t,m(t,\mu))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,m(s,\mu))\,ds\right)} \quad (8)$$

let $x,t,s\in[a,b]$

$$\underline{k(x,t,F_{1\alpha}(x,t,m(t,\mu))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,m(s,\mu))ds\right)}$$
$$= \begin{cases} k(x,t,\underline{F}_{1\alpha}(t,\underline{m}(t,\mu))) \cdot G\left(t,\int_a^t \underline{F}_{2\alpha}(t,s,\underline{m}(s,\mu))ds\right), & \underline{k(x,t,F_{1\alpha}(x,t,m(t,\mu))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,m(s,\mu))ds\right)} \ge 0 \\ k((x,t,\overline{F}_{1\alpha}(t,\overline{m}(t,\mu)) \cdot G\left(t,\int_a^t \overline{F}_{2\alpha}(t,s,\overline{m}(s,\mu))ds\right), & \underline{k(x,t,F_{1\alpha}(x,t,m(t,\mu))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,m(s,\mu))ds\right)} < 0 \end{cases}$$

$$\overline{k(x,t,F_{1\alpha}(x,t,u(t,\alpha))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,u(s,\alpha))ds\right)}$$
$$= \begin{cases} k(x,t,\bar{F}_{1\alpha}(t,\bar{m}(t,\mu))) \cdot G\left(t,\int_a^t \bar{F}_{2\alpha}(t,s,\bar{m}(s,\mu))ds\right), & \overline{k(x,t,F_{1\alpha}(x,t,m(t,\mu))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,m(s,\mu))ds\right)} \ge 0 \\ k(x,t,\underline{F}_{1\alpha}(t,\underline{m}(t,\mu))) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,\underline{m}(s,\mu))ds\right), & \overline{k\left(x,t,F_{1\alpha}(x,t,m(t,\mu))\right) \cdot G\left(t,\int_a^t F_{2\alpha}(t,s,m(s,\mu))ds\right)} < 0 \end{cases} \quad (9)$$

Equation (7) is converted to nonlinear Volterra integral equations for each $0\le\mu\le 1$ and $a\le x\le b$. Now that we've described homotopy analysis approaches as approximate answers $\tilde{m}(x), a\le x\le b$, we find the approximate solutions of $\tilde{m}(x)$.

3.1. HOMOTOPY ANALITIC METHOD "HAM" IN GENERAL

The system (8) is now solved using the homotopy analytic method, and a recursion scheme is obtained. Before HAM for the system (8). We presumptively have four cases for kernels in the kernel.

$$N[\tilde{m}(x,\mu)] = 0$$

$$\tilde{m}(x) = \tilde{f}(x) + \lambda\int_a^x k\left((x,t,\tilde{F}_1(x,t,\tilde{m}(t))) \cdot G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt$$

Where

$\tilde{m}(x,\mu) = (\underline{m}(x,\mu),\bar{m}(x,\mu))$ and $\tilde{f}(x,\mu) = (\underline{f}(x,\mu),\bar{f}(x,\mu))$

As can be seen, equation (7) is transformed into a system of crisp fuzzy Volterra integral nonlinear equations. $\lambda\int_a^x k\left(\left(x,t,\underline{F_1}\left(x,t,\underline{u}(t,\mu)\right)\right)\right) \cdot G\left(t,\int_a^t \underline{F_{2\mu}}\left(t,s,\underline{m}(s,\mu)\right)ds\right)dt$

$$\underline{m}(x,\mu) = \underline{f}(x,\mu) + \lambda\int_a^x k\left(\left(x,t,\underline{F_{1\mu}}\left(x,t,\underline{m}(t,\mu)\right)\right)\right) \cdot G\left(t,\int_a^t \underline{F_\mu}\left(t,s,\underline{m}(s,\mu)\right)ds\right)dt \quad (10)$$

$$\bar{m}(x,\mu) = \bar{f}(x,\mu) + \lambda\int_a^x k((x,t,\overline{F}_{1\mu}(x,t,\overline{m}(t,\mu))) \cdot G\left(t,\int_a^t \overline{F}_{2\mu}(t,s,\overline{m}(s,\mu))ds\right)dt$$

We can see that the homotopy method is used to construct the zero-order deformation equations in order to solve the system (10)

$$(1-p)\mathcal{L}\left[\underline{\emptyset}(x,p;\mu)-\underline{w_0}(x;\mu)\right]=phH\left[\underline{\emptyset}(x,p;\mu)-\underline{f}(x,\mu)\ -\lambda\int_a^x k\left(\left(x,t,\underline{F_{1\mu}}\left(x,t,\underline{\emptyset}(t,p;\mu)\right)\right)\cdot\right.\right.$$
$$G\left(t,\int_a^t \underline{F_{2\mu}}\left(t,s,\underline{\emptyset}(s,p,\mu)\right)ds\right)dt \qquad (11)$$
$$(1-p)\mathcal{L}[\overline{\emptyset}(x,p;\mu)-\overline{w_0}(x;\mu)]=phH\left[\overline{\emptyset}(x,p;\mu)-\overline{f}(x,\mu)\ -\lambda\int_a^x k\left(\left(x,t,\overline{F}_{1\mu}\left(x,t,\overline{\emptyset}(t,p;\mu)\right)\right)\cdot\right.\right.$$
$$G\left(t,\int_a^t \overline{F}_{2\mu}(t,s,\overline{\emptyset}(s,p,\mu))ds\right)dt$$

Where L is an auxiliary linear factor, h is a non-zero auxiliary parameter, and $p\in[0,1]$is the parameter embedding H(x) is a supporting function. $\left(\underline{w_0}(x;\mu),\bar{w}_0(x;\mu)\right)$ are initial guesses $\underline{F_{1\mu}}(x,t,\underline{\emptyset}(t,p;\mu)),\bar{F}_{1\mu}(t,s,\bar{\emptyset}(t,p;\mu)),\bar{F}_{2\mu}(x,t,\bar{\emptyset}(t,p;\mu))$ and $\underline{F_{2\mu}}(t,s,\underline{\emptyset}(t,p;\mu))$ respectively $\emptyset(t,p;\mu)$ and $\bar{\emptyset}(t,p;\mu)$ are unknown functions.. With the zero-order deformation equations previously given and the assumptions that L(m)=m and H(x)=1, we obtain

$$(1-p)\left[\underline{\emptyset}(x,p;\mu)-\underline{w_0}(x;\mu)\right]=$$
$$ph[\underline{\emptyset}(x,p;\mu)-\underline{f}(x,\mu)\ -\lambda\int_a^x k\left((x,t,\underline{F}_{1\mu}\left(x,t,\underline{\emptyset}(t,p;\mu)\right)\right)\cdot$$
$$G\left(t,\int_a^t \underline{F_{2\mu}}\left(t,s,\underline{\emptyset}(s,p,\mu)\right)ds\right)dt \qquad (12)$$
$$(1-p)[\overline{\emptyset}(x,p;\mu)-\overline{w_0}(x;\mu)]$$
$$=ph[\overline{\emptyset}(x,p;\mu)-\overline{f}(x,\mu)\ -\lambda\int_a^x k\left((x,t,\overline{F}_{1\mu}(x,t,\overline{\emptyset}(t,p;\mu))\right)\cdot$$
$$G\left(t,\int_a^t \overline{F}_{2\mu}(t,s,\overline{\emptyset}(s,p,\mu))ds\right)dt$$

It is clear that when p equal (0) and p equal (1), since $h\neq 0$, we get

$$\underline{\emptyset}(x,0;\mu)=\underline{w_0}(x;\mu)\ ,\ \overline{\emptyset}(x,0;\alpha)=\bar{w}_0(x;\mu)$$

And

$$\underline{\emptyset}(x,1;\mu)=\underline{f}(x,\mu)+\lambda\int_a^x k\left((x,t,\underline{F}_{1\mu}(x,t,\underline{\emptyset}(t,1;\mu))).G\left(t,\int_a^t \underline{F_{2\mu}}(t,s,\underline{\emptyset}(s,1,\mu))ds\right)dt\right.$$
$$\bar{\emptyset}(x,1;\mu)=\bar{f}(x,\mu)+\lambda\int_a^x k\left((x,t,\bar{F}_{1\mu}(x,t,\bar{\emptyset}(t,1;\mu))).G\left(t,\int_a^t \bar{F}_{2\mu}(t,s,\bar{\emptyset}(s,1,\mu))ds\right)dt\right. \qquad (13)$$

Respectively. The function $\underline{\emptyset}(x,p;\mu),\bar{\emptyset}(x,p;\mu)$ therefore changes from the initial hypotheses $\underline{w_0}(x;\mu)$, $\bar{w}_0(x;\mu)$ to the solution as p increases from 0 to 1,$\underline{F}_{1\mu}\left(x,t,\underline{\emptyset}(t,p;\mu)\right),\bar{F}_{1\mu}\left(t,s,\bar{\emptyset}(t,p;\mu)\right),\bar{F}_{2\mu}\left(x,t,\bar{\emptyset}(t,p;\mu)\right)$ and$\underline{F_{2\mu}}\left(t,s,\underline{\emptyset}(t,p;\mu)\right)$

Taylor's series expansion $\underline{\emptyset}(x,p;\mu)$ with respect to p results in

$$\underline{\emptyset}(x,p;\mu)=\underline{w_0}(x;\mu)+\sum_{m=1}^{\infty}\underline{u}_m(x,\mu)p^m$$
$$\bar{\emptyset}(x,p;\mu)=\bar{w}_0(x;\mu)+\sum_{m=1}^{\infty}\bar{u}_m(x,\mu)p^m \qquad (14)$$

Where

$$\underline{u}_m(x,r)=\frac{1}{m!}\frac{\partial^m\underline{\emptyset}(t,p;r)}{\partial p^m}\bigg|_{p=0}$$
$$\bar{u}_m(x,r)=\frac{1}{m!}\frac{\partial^m\bar{\emptyset}(t,p;r)}{\partial p^m}\bigg|_{p=0},m\geq 1$$

It is important to remember that $\underline{\emptyset}(x,p;\mu)=\underline{w_0}(x;\mu)$ and $\bar{\emptyset}(x,p;\mu)=\bar{w}_0(x;\mu)$ when the zero-order deformation equations (13) are multiplied by pembedding parameter m times and divided by m! and n! When we eventually put p=0, we obtained what are known as m-th order equations of deformation.

$$\underline{u}_m(x;r)=h\left[\underline{u}_{m-1}(x;r)-(1-x_m)\underline{f}(x;r)-\lambda\int_a^x k\left(\left(x,t,\underline{F_{1\mu}}\left(x,t,\underline{\emptyset_{m-1}}(t,1;\mu)\right)\right)\cdot G\left(t,\int_a^t \underline{F_{2\mu}}\left(t,s,\underline{\emptyset_{m-1}}(s,1,\mu)\right)ds\right)dt\right.\right.$$
$$\bar{u}_m(x;r)=h\left[\bar{u}_{m-1}(x;r)-(1-x_m)\bar{f}(x;r)--\lambda\int_a^x k\left((x,t,\bar{F}_{1\mu}(x,t,\bar{\emptyset}_{m-1}(t,p;\mu))\right)G\left(t,\int_a^t \bar{F}_{2\mu}(t,s,\overline{\emptyset_{m-1}}(s,p,\mu))ds\right)dt\right. \qquad (15)$$

Where

$x_m = \begin{cases} 0, m = 1 \\ 1, m \neq 1 \end{cases} \quad m \geq 1$

$$\underline{\emptyset}(x, \mu) = \underline{w}_0(x; \mu)$$

$$\underline{R}_{m-1}(x, t; \mu) = \frac{1}{(m-1)!} \frac{\partial^{m-1} k\left(x, t, \underline{F}_{1\mu}(x, \underline{\emptyset}(x, p; \mu)) \cdot G\left(t, \int_a^t \underline{F}_{2\mu}(t, s, \underline{\emptyset}(s, p, \mu) ds\right)\right.}{\partial p^{m-1}}\Bigg|_{p=0}$$

and

$$\bar{\emptyset}(x, \mu) = \bar{w}_0(x; \mu)$$

$$\bar{R}_{m-1}(x, t; \mu) = \frac{1}{(m-1)!} \frac{\partial^{m-1} k\left(x, t, \bar{F}_{1\alpha}\left(x, \bar{\emptyset}(x, p; \mu)\right)\right) \cdot G\left(t, \int_a^t \bar{F}_{2\alpha}(t, s, \bar{\emptyset}(s, p, \mu) ds\right)}{\partial p^{m-1}}\Bigg|_{p=0} \tag{16}$$

from equations (15) , (16) we obtain

$$\underline{u}_1(x; \mu) = h\underline{u}_0(x; \mu) - h\underline{f}(x; \mu) - \lambda h\left[\int_0^x \underline{R}_0(x, t; \mu) dt\right]$$
$$\underline{u}_m(x; \mu) = (1 + h)\underline{u}_{m-1}(x; \mu) - \lambda h\left[\int_0^x \underline{R}_{m-1}(x, t; \mu) dt\right]$$

Similarly

$$\bar{u}_1(x; \mu) = h\bar{u}_0(x; \mu) - h\bar{f}(x; \mu) - \lambda h\left[\int_0^x \bar{R}_0(x, t; \mu) dt\right]$$
$$\bar{u}_m(x; \mu) = (1 + h)\bar{u}_{m-1}(x; \mu) - \lambda h\left[\int_0^x \bar{R}_{m-1}(x, t; \mu) dt\right] \tag{17}$$

as,

$$\underline{R}_0(x, t; \mu) = k\left(x, t, \underline{F}_{1\mu}\left(t, \underline{u}_0(t, \mu)\right)\right) \cdot G\left(t, \int_a^t \underline{F}_{2\mu}\left(t, s, \underline{u}_0(s, \mu)\right) ds\right)$$
$$\bar{R}_0(x, t; \mu) = k\left(x, t, \bar{F}_{1\mu}\left(t, \bar{u}_0(t, \mu)\right)\right) \cdot G\left(t, \int_a^t \bar{F}_{2\mu}\left(t, s, \bar{u}_0(s, \mu)\right) ds\right) \tag{18}$$

If we select the values of h properly, the series (9) is convergent of p=1. The homotopy solution series, system solution (10) as shown below:

$$\underline{U}(x, \mu) = \underline{u}_0(x; \mu) + \sum_{m=1}^{\infty} \underline{u}_m(x; \mu)$$
$$\bar{U}(x, \mu) = \bar{u}_0(x; \mu) + \sum_{m=1}^{\infty} \bar{u}_m(x; \mu) \tag{19}$$

We give the n^{th} order solution approximately with

$$\underline{U_j}(x, r) = \underline{u}_0(x; r) + \sum_{m=1}^{j} \underline{u}_m(x; r)$$
$$\overline{U_j}(x, r) = \bar{u}_0(x; r) + \sum_{m=1}^{j} \bar{u}_m(x; r), \; j \geq 1 \tag{20}$$

3.2. ANALISYSIS OF EXISTENCE AND CONVERGENCE:

Consider $\tilde{f}(x)$ to be bounded $\forall x \in [a, b]$ and

$|k(g, h)| \leq \Re, \forall a \leq g, h, \leq b$

We also assume that the nonlinear factors $\tilde{F}_1(g, h, \tilde{m}(h)), \tilde{F}_2(h, g, \tilde{m}(g))$ are satisfied in Lipschitz condition

$$D\left(\tilde{F}_1(g, h, \tilde{m}(h)) - \tilde{F}_1(g, h, \tilde{m}^*(h))\right) \leq LL_1 D(\tilde{m}, \tilde{m}^*)$$
$$D\left(\tilde{F}_2(h, g, \tilde{m}(g)) - \tilde{F}_2(h, g, \tilde{m}^*(g))\right) \leq L^* L_1^* D(\tilde{m}, \tilde{m}^*)$$

Let

$\alpha = \lambda(LL_1\Re + L^* L_1^* \Re^*)$

Using the Homotopy analysis method, we will now demonstrate the existence, uniqueness, and convergence of the method's solution.

Theorem 3.2.1: For $0 < \mu < 1$, then we get a unique solution for equation (7).

Proof: let m and m^* get two solutions of equation (7).

$$D(\tilde{m}, \tilde{m}^*) = D(\tilde{f}(x) + \lambda \int_a^x k\left(\left(x, t, \tilde{\tilde{F}}_1(x, t, \tilde{m}(t))\right) \cdot G\left(t, \int_a^t \tilde{\tilde{F}}_2(t, s, \tilde{m}(s)) ds\right) dt\right.$$

$$\tilde{f}(x) + \lambda \int_a^x k\left(\left(x, t, \tilde{\tilde{F}}_1(x, t, \tilde{m}^*(t))\right) \cdot G\left(t, \int_a^t \tilde{\tilde{F}}_2(t, s, \tilde{m}^*(s)) ds\right) dt\right)$$

$$D(\tilde{m}, \tilde{m}^*) = max\{sup \mid \tilde{f}(x) + \lambda \int_a^x k\left((x, t, \tilde{F}_1(x, t, \tilde{m}(t))\right) \cdot G\left(t, \int_a^t \tilde{F}_2(t, s, \tilde{m}(s)) ds\right) dt$$

$$-\tilde{f}(x) + \lambda \int_a^x k\left(\left(x, t, \tilde{F}_1(x, t, \tilde{m}^*(t))\right) G\left(t, \int_a^t \tilde{F}_2(t, s, \tilde{m}^*(s)) ds\right) dt\right.$$

$$\mid \; sup \mid \tilde{f}(x) + \lambda \int_a^x k\left(\left(x, t, \tilde{F}_1(x, t, \tilde{m}^*(t))\right) \cdot G\left(t, \int_a^t \tilde{F}_2(t, s, m^*(s)) ds\right) dt\right.$$

$$-\tilde{f}(x)+\lambda\int_a^x k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}(s))ds\right)dt\right.$$

$$\leq max\{s\,u\,p\,\lambda(\int_a^x |\,k(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}(s))ds\right)dt$$

$$-\int_a^x k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}^*(s))ds\right)dt\,|\right),$$

$$sup\lambda(\int_a^x |\,k(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}^*(s))ds\right)dt$$

$$|\,-\int_a^x k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}(s))ds\right)dt\,|\right),$$

$$\leq max\{s\,u\,p\,\lambda(\int_a^x |\,k(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}(s))ds\right)dt-k(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)$$

$$\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}^*(s))ds\right)+k(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}^*(s))ds\right)$$

$$-k(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2\,(t,s,\tilde{m}^*(s))ds\right)dt\,|\},$$

$$sup\lambda(\int_a^x |\,k(\left(x,t,F_1(x,t,\tilde{m}^*(t))\right)\cdot G\left(t,\int_a^t F_2\,(t,s,\tilde{m}^*(s))ds\right)dt$$

$$-k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt\right.$$

$$+k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt\right.$$

$$-k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)\cdot G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt\,|\right\}$$

$$\leq \max\left\{\sup|\,\lambda\int_a^x k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)dt\right|\cdot|G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)-\right.$$

$$\left|G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}^*(s))ds\right)\right|+\left|G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}^*(s))ds\right)\right|\cdot|\,k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)-k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\right|,\right.$$

$$\sup|\,\lambda\int_a^x k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\right|\cdot|G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}^*(s))ds\right)-G\left(t,\int_a^t F_2(t,s,\tilde{m}(s))ds\right)|+$$

$$\left|G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)\right|.|\,k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t)\right)-k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)\right|\right\}$$

$$\leq \lambda\sup_{a<x<b,0<t<1}|\,D\left(k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right),\tilde{0}\right)\right|\cdot|\int_a^x G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt-\int_a^x G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}^*(s))ds\right)dt\,|$$

$$+\left|\int_a^x G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}^*(s))ds\right)dt\right|.D\,|\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right)-k\left(\left(x,t,\tilde{F}_1(x,t,\tilde{m}^*(t))\right)\right|,\right.$$

$$\lambda \sup_{a<x<b, 0<t<1} D \mid \left(k\left((x,t,\tilde{F}_1(x,t,\tilde{m}^*(t)),\eth)\right| \cdot \left|\int_a^x G\left(t, \int_a^t \tilde{F}_2(t,s,\tilde{m}^*(s))ds\right)dt\right| - \int_a^x G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt\right|$$

$$+\left|\int_a^x G\left(t,\int_a^t \tilde{F}_2(t,s,\tilde{m}(s))ds\right)dt\right| \cdot D \mid k\left((x,t,\tilde{F}_1(x,t,\tilde{m}^*(t)) - k\left((x,t,\tilde{F}_1(x,t,\tilde{m}(t))\right) \mid\right.\right.$$

$$\leq \lambda \sup\left(\int_a^x D(\tilde{F}_1(x,t,\tilde{u}(t)),\eth) \cdot L\left(\int_a^x D\left(\tilde{F}_2(t,s,\tilde{u}(s)) - \tilde{F}_2(t,s,\tilde{m}^*(s))\right)ds\right) + \left(\int_a^x D\left(\tilde{F}_2(t,s,\tilde{m}^*(s))\right)ds,\eth\right).\right.$$

$$L_1\left(\tilde{F}_1(x,t,\tilde{m}(t)) - \tilde{F}_1(x,t,\tilde{m}^*(t))\right), \lambda\sup\left(D(\tilde{F}_1(x,t,\tilde{m}^*(t)),\eth) \cdot L^*\left(\int_a^x D(\tilde{F}_2(t,s,\tilde{u}^*(s)) - \tilde{F}_2(t,s,\tilde{m}(s))ds\right)\right.$$

$$+\left(\int_a^x D(\tilde{F}_2(t,s,\tilde{m}(s))ds\right) \cdot L_1^*\left(\tilde{F}_1(x,t,\tilde{m}(t)) - \tilde{F}_1(x,t,\tilde{m}^*(t))\right)$$

$$\leq \lambda \sup D(\tilde{F}_1(x,t,\tilde{u}(t)),\eth) \cdot LL_1\left(\tilde{F}_2(t,s,\tilde{u}^*(s)) - \tilde{F}_2(t,s,\tilde{u}(s))\right), \lambda\sup D(\tilde{F}_2(t,s,\tilde{u}^*(s)),\eth) \cdot L^*L_1^*\left(\tilde{F}_1(x,t,\tilde{u}(t)) - \tilde{F}_1(x,t,\tilde{u}^*(t))\right)$$

$$\leq \lambda \sup LL_1 D(\tilde{F}_1(x,t,\tilde{m}(t)),\eth) \cdot D(\tilde{m},\tilde{m}^*), \lambda\sup L^*L_1^* D(\tilde{F}_2(x,t,\tilde{m}(t)),\eth) \cdot D(\tilde{m},\tilde{m}^*)$$

$$\leq \lambda \sup(b-a)\, LL_1 D(\tilde{F}_1(x,t,\tilde{u}(t)),\eth) \cdot D(\tilde{m},\tilde{m}^*), \lambda \sup(b-a)\, L^*L_1^* D(\tilde{F}_2(x,t,\tilde{m}(t)),\eth) \cdot D(\tilde{m},\tilde{m}^*)$$

$$\leq \lambda(b-a) LL_1 D(\tilde{F}_1(x,t,\tilde{m}(t)),\eth) \cdot D(\tilde{m},\tilde{m}^*), \lambda(b-a) L^*L_1^* D(\tilde{F}_2(x,t,\tilde{m}(t)),\eth) \cdot D(\tilde{m},\tilde{m}^*)$$

Then we let $D(\tilde{F}_1(x,t,\tilde{m}(t)),\eth) = \Re$ and $D(\tilde{F}_2(x,t,\tilde{m}(t)),\eth) = \Re^*$

We get

$\leq \lambda(LL_1\Re + L^*L_1^*\Re^*)(b-a)^2 D(\tilde{m},\tilde{m}^*)$

let

$\alpha = \lambda(LL_1\Re + L^*L_1^*\Re^*) = \alpha D(\tilde{m},\tilde{m}^*)$

From which we get $(1-\mu)D(\tilde{m},\tilde{m}^*) \leq 0$. Since $0<\mu<1$, then $D(\tilde{m},\tilde{m}^*) = 0$. Implies $\tilde{m} = \tilde{m}^*$

Theorem 3.2.2. Consider the series $\bar{m}(x,\mu) = \sum_{i=0}^{\infty} \bar{m}_i(x,\mu)$ of equation (7) using HAM convergence when $0<\mu<1, |\bar{m}_i(x,\mu)| < \infty$

Proof: let $\bar{T}_n$ and $\bar{T}_m$ be any two partial sums with $n \geq m$. In the series of partition sums $\bar{T}_n$ In this Banach space, we will demonstrate that is a Cauchy sequence.

$$||\bar{T}_n - \bar{T}_m|| = \max_{\forall x\in[a,b]} \mid \lambda\left[\int_a^x k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{n-1}(h))\right) \cdot G\left(h,\int_a^t \tilde{F}_2(h,g,\tilde{m}_{n-1}(g))dg\right)dh\right.$$

$$-\left[\int_a^x k\left((x,t,\tilde{F}_1(x,h,\tilde{m}_{m-1}(h))\right) \cdot G\left(h,\int_a^t \tilde{F}_2(h,g,\tilde{m}_{m-1}(g))dg\right)dh\right.$$

$$\leq \int_\mu^x \mid k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{n-1}(h))\right) \cdot G\left(h,\int_a^t \tilde{F}_2(h,s,\tilde{m}_{n-1}(g))dg\right)dh - k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{n-1}(h))G\left(h,\int_a^t \tilde{F}_2(h,g,\tilde{m}_{m-1}(g))dg\right.\right.$$

$$+k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{n-1}(h))G\left(h,\int_a^t \tilde{F}_2(h,g,\tilde{m}_{m-1}(g))dg\right.\right.$$

$$+\int_a^x k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{m-1}(h))\right) \cdot G\left(h,\int_a^t \tilde{F}_2(t,g,\tilde{u}_{m-1}(g))dg\right)dh\Big] \mid,$$

$$\int_a^x \mid k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{m-1}(h))\right) \cdot G\left(h,\int_a^h \tilde{F}_2(h,g,\tilde{m}_{m-1}(g))dg\right)dh - k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{m-1}(h))G\left(h,\int_a^h \tilde{F}_2(t,g,\tilde{m}_{n-1}(g))dg\right.\right.$$

$$+k\left((x,h,\tilde{F}_1(x,t,\tilde{m}_{m-1}(h))G\left(h,\int_a^t \tilde{F}_2(h,g,\tilde{m}_{n-1}(g))dg + k\left((x,h,\tilde{F}_1(x,h,\tilde{m}_{n-1}(h))\right).G\left(h,\int_a^h \tilde{F}_2(t,g,\tilde{u}_{n-1}(g))dg \mid\right.\right.\right.$$

We get
$\leq \mu||\bar{T}_n - \bar{T}_m||$
Let $n = m+1$, then

$||\bar{T}_n - \bar{T}_m|| \leq \mu||\bar{T}_m - \bar{T}_{m-1}|| \leq \alpha^2||\bar{T}_{m-1} - \bar{T}_{m-2}|| \ldots\ldots\ldots\ldots\ldots \leq \mu^m||\bar{T}_1 - \bar{T}_0||$
We have,

$$||\bar{T}_n - \bar{T}_m|| \leq \quad ||\bar{T}_{m+1} - \bar{T}_m|| + ||\bar{T}_{m+2} - \bar{T}_{m+1}|| + \cdots \dots \dots \dots + ||\bar{T}_n - \bar{T}_{n-1}||$$
$$\leq [\mu^m + \mu^{m+1} + \cdots \dots \dots + \mu^{m-1}]||\bar{T}_1 - \bar{T}_0||$$
$$\leq \alpha^m[1 + \mu + \mu^2 + \cdots \dots \dots . + \mu^{n-m-1}]||\bar{T}_1 - \bar{T}_0||$$
$$\leq \alpha^m \left[\frac{1 - \mu^{n-m}}{1 - \mu}\right] ||\bar{m}_1(x, \mu) \|$$

Since $0 < \mu < 1$, we have $(1 - \mu^{n-m}) < 1$, then

$$||\bar{T}_n - \bar{T}_m|| \leq \frac{\mu^m}{1 - \mu} \max_{\forall t} |\bar{m}_1(x, \mu)|$$

But $|\bar{m}_1(x, \mu)| < \infty$, as $m \to \infty$, then $\| \bar{T}_n - \bar{T}_m|| \to 0$. We conclude that $\bar{T}_n$ is Cauchy sequence

$$\bar{m}(x, \mu) = \lim_{n\to\infty} \bar{m}_n(x, \mu)$$

$\underline{T}_n$ is a Cauchy sequence, therefore similarly, we have

$$\underline{m}(x, \mu) = \lim_{n\to\infty} \underline{m}_n(x, \mu)$$

Finally we have

$$\tilde{m}(x, \mu) = \lim_{n\to\infty} \tilde{m}_n(x, \mu)$$

Theorem 3.2.3. In nonlinear fuzzy when using the homotopy analysis method, the Volterra integral equation converges to the exact solution.

For $n \geq 0$

$$\vartheta_{n+1}(g, h) = \tilde{f}(x) + \sum_{i=1}^{n+1} \left[\lambda \int_a^x k\left(\left(x, h, \tilde{F}_1(x, h, \tilde{m}_i(h))\right) \cdot G\left(h, \int_a^h \tilde{F}_2(h, g, m_h(g))dg\right) dh\right]$$

$$D(\vartheta_{n+1}(x, h), \vartheta_n(x, h))$$

$$= D\left(\vartheta_n(x, h) + \lambda \int_a^x k\left(\left(x, h, \tilde{F}_1(x, h, \tilde{m}_{n+1}(h))\right) \cdot G\left(h, \int_a^h \tilde{F}_2(h, g, \widetilde{m_{n+1}}(g))dg\right) dh, \vartheta_n(x, h)\right)\right)$$

$$= D\left(\lambda \int_a^x k\left(\left(x, h, \tilde{F}_1(x, h, \tilde{m}_{n+1}(h))\right) \cdot G\left(h, \int_a^h \tilde{F}_2(h, g, \widetilde{m_{n+1}}(g))dg\right) dh, \tilde{0}\right)\right) \leq D(\tilde{m}_{n+1}(x), \tilde{0})$$

$$D(\tilde{m}_n(x), \tilde{0}) \leq \mu^n D(\tilde{f}, \tilde{0})$$

So

$$D(\vartheta_{n+1}, \vartheta_n) \leq \mu^{n+1} D(\tilde{f}, \tilde{0})$$

Then

$$\sum_{n=0}^{\infty} D(\vartheta_{n+1}, \vartheta_n) \leq \mu^{n+1} D(f, 0) \sum_{n=0}^{\infty} \mu^n, 0 < \mu < 1$$

3.3. NUMERICAL EXAMPLE

Consider the nonlinear fuzzy Volterra integral using the homotopy analysis method calculate the absolute error between the approximation of homotopy analysis method and the exact solution.

$$\tilde{m}(x) = \tilde{f}(x) + \int_a^x k\,(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{m}(t))dt$$

We get

$$\underline{m}(x, \mu) = \underline{f}(x, \mu) + \int_a^c k\,(x, t, \underline{k_1}\left(t, \underline{F}(t, m(t, \mu))\right) + \int_c^x k\,(x, t, \overline{k_1}\left(t, \overline{F}(t, \overline{m}(t, \mu))\right)$$

$$\overline{m}(x, \mu) = \overline{f}(x, \mu) + \int_a^c k\,(x, t, \overline{k_1}\left(t, \overline{F}(t, \overline{m}(t, \alpha))\right) + \int_c^x k(x, t, \underline{k_1}\left(t, \underline{F}\left(t, \underline{m}(t, \mu)\right)\right)$$

Then we obtain

$$\underline{m}(x, \mu) = \underline{f}(x, \mu) + \int_a^d k(x, t, \underline{k_1}(t, \underline{F}\left(t, \underline{m}(t, \mu)\right) dt + \int_d^c k(x, t, \overline{k_1}(\overline{F}(t, \overline{m}(t, \mu))dt + \int_c^e k(x, t, \underline{k_1}(t, \left(t, \overline{m}(t, \mu)\right)dt$$
$$+ \int_e^x k(x, t, \overline{k_1}(t, \underline{F}\left(t, \underline{m}(t, \mu)\right)\; dt$$

$$\overline{m}(x, \mu) = \overline{f}(x, \mu) + \int_a^d k(x, t, \overline{k_1}(\overline{F}(t, \overline{m}(t, \mu))dt + \int_d^c k(x, t, \underline{k_1}(t, \underline{F}\left(t, \underline{m}(t, \mu)\right) dt$$
$$+ \int_c^e k(x, t, \overline{k_1}(t, \underline{F}\left(t, \underline{m}(t, \mu)\right) dt + \int_e^x k(x, t, \underline{k_1}(t, \overline{F}(t, \overline{m}(t, \mu))dt$$

Where $a \leq t \leq d, d \leq t \leq c, c \leq t \leq e, e \leq t \leq x, 0 \leq \mu \leq 1$

Kernal is

$k(x,t,\underline{k_1}(t,\underline{F}(t,\underline{m}(t,\mu)) = x\text{^}2t(1-2t)(\underline{m}(t,\mu))\text{^}2$

$k(x,t,\overline{\overline{k_1}}(\overline{F}(t,\overline{m}(t,\mu)) = x\text{^}2(1-2t)(1-t)\text{^}2(\overline{m}(t,\mu))\text{^}3$

$k(x,t,\underline{k_1}(t,\overline{F}(t,\overline{m}(t,\mu)) = x\text{^}2t(1-2t)(\overline{m}(t,\mu))\text{^}3$

$k(x,t,\overline{\overline{k_1}}(t,\underline{F}(t,\underline{m}(t,\mu)) = x\text{^}2(1-2t)(1-t)\text{^}2(\underline{m}(t,\mu))\text{^}2$

$a = 0, d = \frac{1}{4}, c = \frac{1}{2}, e = \frac{3}{4}, x = 1$

The exact solution

$\underline{m}(x,\mu) = \left(\frac{5-\mu}{4}\right) exp(x)\ , \overline{m}(x,\mu) = \left(\frac{3+\mu}{4}\right) exp(x)$

$\underline{f}(x,\mu) = (5/4-(1/4)^*r)^* exp(x) + .2620498398^*r\text{^}3^*x\text{^}5 - 2.225551737^*r^*x\text{^}2 + 5.882874723^*x\text{^}2$
$- 0.2799479167e - 3^*x\text{^}3^*(.7500000000 + .2500000000^*r)\text{^}3^*(exp(x))\text{^}3 + 0.3375612483e$
$- 2^*r\text{^}3^*x\text{^}4 + (1/14)^*exp(2^*x)^*r\text{^}2^*x\text{^}5 - (1/4)^*exp(2^*x)^*r\text{^}2^*x\text{^}4 - (5/8)^*exp(2^*x)^*r^*x\text{^}5$
$+ (3/8)^*exp(2^*x)^*r\text{^}2^*x\text{^}3 + (5/2)^*exp(3^*x)^*r^*x\text{^}4 + (25/16)^*exp(2\ ^*x)^*x\text{^}5$
$- (7/32)^*exp(3^*x)^*r\text{^}3^*x\text{^}4 - (15/4)^*exp(2^*x)^*r^*x\text{^}3 - (25/4)^*exp(2^*x)^*x\text{^}4$
$+ (35/16)^*exp(22^*x)^*r^*x\text{^}2 + (75/8)^*exp(22^*x)^*x\text{^}3 - (175/32)^*exp(2^*x)^*x\text{^}2$

$\overline{f}(x,\mu) = (3/4+(1/4)^*r) * exp(x) - 0.7161458333e - 2 * x\text{^}2$
$* (1.250000000 -.2500000000^*r)\text{^}2^*(exp(x))\text{^}2 - (63/64)^*exp(3^*x)^*x\text{^}4$
$+ (39/32)^*exp(3^*x)^*x\text{^}3 - (35/64)^*exp(3^*x)^*x\text{^}2 + (39/32)^*exp(3^*x)^*r^*x\text{^}3$
$- (35/64)^*exp(3^*x)^*r^*x\text{^}2 + (9/32)^*exp(3^*x)^*x\text{^}5 + (13/32)^*exp(3^*x)^*r\text{^}2^*x\text{^}3$
$- (63/64)^*exp(4*x)^*r^*x\text{^}3 - (35/192)^*exp(3^*x)^*r\text{^}2 * x\text{^}2 + (13/288)^*exp(3^*x)^*r\text{^}3 * 3\text{^}3$
$- (21/64)^*exp(3^*x)^*r\text{^}2 * x\text{^}3 + (9/32)^*exp(3^*x)^*r^*x\text{^}5 - (35/1728)^*exp(3^*x)^*r\text{^}3^*x\text{^}2$
$+ (1/96)^*exp(3^*x)^*r\text{^}3^*x\text{^}5 - (7/192)^*exp(3^*x)^*r\text{^}3^*x\text{^}4 + (3/32)^*exp(3^*x)^*r\text{^}2^*x\text{^}5$
$+ 0.2063574710e - I^*r\text{^}3^*x\text{^}2 + .1925350109^*r\text{^}3^*x\text{^}4 + .4890323008^*r^*x\text{^}2$
$+ .7274973486^*x\text{^}2$

HAM is now used to resolve this issue.

$\underline{m}_0(x,\mu) = \underline{f}(x,\mu)\ , \overline{m}_0(x,\mu) = \overline{f}(x,\mu)$

$\underline{m}_1(x;\mu) = h\underline{m}_0(x;\mu) - h\underline{f}(x;\mu) - \lambda h[\int_0^x \underline{R}_0(x,t;\mu)dt]$

$\underline{m}_m(x;\mu) = (1+h)\underline{m}_{m-1}(x;\mu) - \lambda h[\int_0^x \underline{R}_{m-1}(x,t;\mu)dt]$

$\overline{m}_1(x;\mu) = h\overline{m}_0(x;\mu) - h\overline{f}(x;\mu) - \lambda h[\int_0^x \overline{R}_0(x,t;\mu)dt]$

$\overline{m}_m(x;\mu) = (1+h)\overline{m}_{m-1}(x;\mu) - \lambda h[\int_0^x \overline{R}_{m-1}(x,t;\mu)dt]$

$\underline{m}_1(x,\mu) =$

$1.874381717 * exp(2..^*x)^*x\text{^}8 + .3299679124^*exp(7.\ ^*x)^*x\text{^}2 + 52.81171591 * exp(7.*x)^*x\text{^}8 -$
$287.0736043^*exp(6*^*x)^*x\text{^}9 + .7586009462^*exp(5*^*x)^*x\text{^}10 - 15.40620635^*exp(3.\ ^*x)^*x\text{^}12 +$
$9.347190553^*exp(9.\ ^*x)^*x\text{^}5 - .3710637270^*exp(8.^*x)^*x\text{^}6 - 46.08017334^*exp(7.*^*x)^*x\text{^}7 +$
$335.1001136^*exp(6.\ ^*x)^*x\text{^}8 - .1080861379 * exp(5.*x)^*x\text{^}9 - 15.74782141^*exp(4.^*x)^*x\text{^}10 -$
$176.7206816^*exp(3.^*x)^*x\text{^}10 - 140.7045353^*exp(6*^*x)^*x\text{^}11 + 13.38390445^*exp(2.*x)^*x\text{^}5 -$
$14.17087252 * exp(4..^*x)^*x\text{^}2 - 19.46203079 * exp(2..^*x)^*x\text{^}4 + 19.46203079 * exp(2.\ ^*x)^*x\text{^}3 -$
3.045033166*
$exp(7..^*x)^*x\text{^}13 + 14.34338461^*exp(6.^*x)^*x\text{^}14 - 81.88951554 * exp(9..^*x)^*x\text{^}10 +$
$.7584369853^*exp(8..^*x)^*x\text{^}11 + 9.921254546^*exp(7.\ ^*x)^*x\text{^}12 - 38.08278113^*exp(6.^*x)^*x\text{^}13 +$
$72.98486976 * exp(9.^*x)^*x\text{^}9 - .8926411169 * exp(8.^*x)^*x\text{^}10$

$\underline{m}_2(x,\mu) =$

$14695.44943^*r\text{^}3^*exp(2.^*x)^*x\text{^}3 + 3.977103959^*r\text{^}4^*exp(3.\ ^*x)^*x\text{^}3 + 318.1620015^*r\text{^}3^*exp(4.\ ^*x)^*x\text{^}3 +$
$4772.436337\ ^*r\text{^}2^*exp(4$
.
$^*x)^*x\text{^}4 - 0.4219515604e - 3^*exp(2.\ ^*x)^*r\text{^}5^*x\text{^}12 + 0.3157028308e - 4^*exp(6*^*x)^*r\text{^}4^*x\text{^}8$
$- 0.3444030882e -$

2*exp(6.*x)*r^4*x^9 + 0.3346252442e − 2*exp(5.*x)*r^3*x^11 − 0.8610077204e − 5*exp(6.*x)*r^4*x^9
+ 0.7064310710e − 2*exp(5. *x)*r^3*x^11 − 0.3936767579e − 3*exp(5. *x)*r^2*x^12
− 13.42773438 ∗ exp(4. *x)*r*x^14 + 0.6672033692e − 1*r^3*exp(5. *x)*x^9
− 0.1310943604e − 1 *r^2*exp(5.*x)*x^10 − 0.3996776579e − 5*r^5*exp(6. *x)*x^5
+ 0.4496373651e − 4*r^4*exp(6. *x)*x^6 − 0.1951617500e − 3*r^3*exp(6 *x)*x^7
+ 0.5995164868e − 5*r^5*exp(6. *x)*x^6 − 0.4879043749e − 4*r^4*exp(6. *x)*x^7
− 0.3765526937e − 2*r^5*exp(5.*x)*x^7 − 0.2955681356e − 2*r^4 ∗ exp(5. *x)*x^8
+ 7954.104768*r*exp(4.∗*x)*x^3 + 1.307137966*r^3*exp(3. *x)*x^2 + 57.73784441*r^2
∗ exp(3.∗x)*x^3 + 431.5076351*r*exp(3 ∗* x)*x^4 − 0.1755798340e
− 2*r^5*exp(5 ∗* x)*x^9 − 0.7283020020e − 3*r^4*exp(5. *x)*x^10 + 0.5915671419e
− 1*r*exp(5.∗* x)*x^2 + 596.5545421*r^2*exp(4. *x)*x^2 + 0.7283020020e
− 3*r^5*exp(5. *x)*x^10 +.7664794922*r^4*exp(4.*x)*x^12
+ 3.564453125*r^3*exp(4. *x)*x^13 + 704.5607766*r^4*exp(2. *x)*x^3
+ 14695.44943*r^3*exp(2. *x)*x^4 + 71372.78381*r^2*exp(2. *x)*x^5 − 5.208912897
∗ r^5*exp(2. *x)*x^2 − 121.9467163*r^3*exp(4* *x)*x^10
− 364.0411377*r^2*exp(4.*x)*x^11 − 383.2397461*r*exp(4.*x)*x^12 − 0.2943112134e
− 1*r^5*exp(2.*x ∗* x^10 − 12.30883418*r^2*exp(3. *x)*x^8
1.311608598*r^5*exp(2. *x)*x^6 + 93.92146117*r^4*exp(2.∗x)*x^7 + 651.0081789*r^3*exp(2. *x)*x^8
+ 0.4535436908e − 2 *r^5 ∗ exp(3.∗* x)*x^5 − 0.2606035500e − 2*r^4*exp(3.*x)*x^6
− 2.251201753*r^3*exp(3. *x)*x^7 + 21210.93818*r*exp(4.∗x)*x^5 + 0.3490261830e
− 4*r^6*exp(3. *x)*x^3 − 0.5235392746e − 4*r^6*exp(3. *x)*x^4
−.7110983277*r*exp(5. *x)*x^9 − 33.92793147*r^4*exp(4. *x)*x^7
− 451.1860657*r^3*exp(4. *x)*x^8 − 1909.107971*r^2*exp(4.*x)*x^9
− 3048.667908*r*exp(4. *x)*x^I0 − 1958.976612*r^3*exp(2. *x)*x^7
= 4742.723957*r^2*exp(2. *x)*x^8 − 4378.383430*r*exp(2. *x)*x^9
− 12.72738647*r^4*exp(4.∗* x)*x^9 + 1213.470459*r*exp(4. *x)*x^11 + 0.2296864505e
− 4*r^6*exp(3.∗x)*x^7 − 0.3043040853e − 2*r^5*exp(5.∗)*x^5 − 0.3803801067e − 2*r^4
∗ exp(5.∗* x)*x^6 + .1430900236 ∗ r^3*exp(5. *x)*x^7 − 0.5320226441e
− 1*r^2*exp(5.*x)*x^8 − 0.2997582434e − 4*r^4*exp(6.*x)*x^5 + 0.1798549460e
− 3*r^3*exp(6.*x)*x^6 − 0.4391139374e − 3*r^2*exp(6. *x)*x^7
+.1291171775*r^5*exp(2.*x)*x^9 − 1.990435827*r^4*exp(2.*x)*x^10 + 0.1998388289e
− 5*r^5*exp(6.*x)*x^4 + 0.4856083543e − 3*r*exp(6 ∗* x)*x^6 − 0.7303298048e
− 3*r^5*exp(5. *x)*x^3 − 0.1825824512e − 2*r^4*exp(5. *x)*x^4
+.1156355524*r^3*exp(5 ∗* x)*x^5 + 31163420610e − 4*r^6*exp(3 ∗* x)*x^2
− 0.4995970724e − 5*r^4*exp(6.*x)*x^3 − 25995164868e − 4*r^3*exp(6 *x)*x^4
− 0.2697824191e − 3*r^2*exp(6.∗x)*x^5 + 95.890585588*r*exp(3. *x)*x^2 − 53529.41599
∗ r^3 ∗ exp(5..*x)*x^3 + 7347.724713*r^3*exp(2. *x)*x^2 − 1.40020506516e
− 2*r^3*exp(5. *x)*x^3 − 0.1314593649e −
1*r^3*x^9 +.1182353533*r^2*x^9 +.1109235065*r*x^9 − 4023.662933 ∗ exp(x)*r^2*x^5
+ 1180.760646*exp(x)*r*x^6 + 4149.831183*exp(x)*r^4 ∗ x^2 −

12492.20079* exp(x)*r^3 * *x^3 + 12070.98880*exp(x)*r^2*x^4 − 4723.042584 * exp(x)*r*x^5 −
74.14719453*r^6*exp(3..* x)*x^8 + 140.5186870*r^5*exp(3. *x)*x^9
− 158.6186204*r^4*exp(3..* x)*x^10 + 152.5983335*r^2 * exp(3. *x)*x^11
+ 0.1809742766e − 3*r^8*exp(2. *x)*x^7 − 0.5224328193e
− 3*r^7*exp(2.** x)*x^8 − 0.4658107994e − 3*r^6*exp(2. *x)*x^9
− 65.43033387*r^3*exp(9. *x)*x^6 + 151.4244870*r^2*exp(9. *x)*x^7
= 170.3525478*r*exp(9. *x)*x^8 − 327.9629648*r^5*exp(3.** x)*x^8
+ 424.6006600*r^4*exp(3. *x)*x^9 − 332.0391579*r^3*exp(3. *x)*x^10
− 1.192977517*10^(−7)*r^8*exp(8. *x)*x^2 +. 1314378641*exp(x)*r^6*x^8
− .1216371704 * exp(x)*r^5*x^9 + 1.155553119*exp(x)*r^5*x^8
−. 5851425808*exp(x)*r^4*x^9 + 5.558854518*exp(x)*r^4*x^8
− 1.761449633*exp(x)*r^3*x^9 + 16.73377151*exp(x)*r^3*x^8
− 3.404114157*exp(x)*r^2*x^9 + 32.33908449 * 9*exp(x)*r^2*x^8
− 3.995805908 * exp(x)*r*x^9 + 37.96015613*exp(x)*r*x^8 − .1321311429
* exp(88*x)*x^14 − 70.34197405*exp(9.* x)*x^12 + .3065881334
* exp(8.* x)*x^13 − 3.937117609*exp(6. *x)*x^15 + 80.94375884 * exp(9.
* x)*x^11 −. 5388782263*exp(8.* x)*x^12 + 120.5276894*exp(4.* x)*x^7
− 1104.209279 * exp(3.* x)*x^8 +. 5329995405*exp(2.* x)*x^9
− 0.2174201525e − 2*exp(8.** x)*x^2 − 2.309775387*exp(7..* x)*x^3
+ 93.09283540*exp(6.* x)*x^4 + 9.147654733*exp(5. *x)*x^5
− 151.1307965*exp(4.** x)*x^6 + 2208.467401 * exp(3.* x)*x^7

$\overline{m}_1(x, \mu)$=
56220011*exp(4. *x)*r^2 *x^15 + 1.3741120848*r^3*exp(3. *x)*x^4 + 86.05989162*r^2*exp(3. *x)*x^5 +
0.231647600e = 2*r^4*exp(3.** x)*x^3 − 0.1007874868e − 2*r^5*exp(3.** x)*x^2 − 0.1148010294e −
5*exp(6. *x)*r^5*x^9 + 0.1859029134e − 3*exp(5. *x)*r^4*x^11 − 0.1859029134e −
3*exp(5.* x)*r^5*x^11 − 0.2187093099e − 4*exp(5.* x)*r^4*x^12 −. 5371093750*exp(4.* x)*r^3*x^14 −
0.1953125000e − 2*exp(4. *x)*r^4*x^15 + 0.2187093099e − 4*exp(5. *x)*r^5*x^12 + 0.2685546875e −
1*exp(4. *x)*r^4*x^14 + 0.3906250000e − 1*exp(4..* x)*r^3*x^15 + 2.864543278*exp(3. *x)*r^2*x^9 +
1.562500000*exp(3. *x)*r*x^10 − 164.5470897*exp(2. *x)*r*x^11 − 14.26532454 * exp(3. *x)*r*x^9 −
0.8310953777e − 3*exp(5.**)*r^3*x^12 + 4.028320312*exp(4.** x)*r^2 * x^14 + .9765625000 * exp(4.*
x)*r*x^15 + 0.4209371078e − 5*exp(6. *x)*r^5*x^8 +. 2949623108*r*exp(5. *x)*x^10 +
33.05508825*r^2*exp(3.* x)*x^7 + 61.26677696 *r*exp(3.* x)*x^8 −. 1782226562*r^4*exp(4.* x)*x^13 +
31.81683167*r^4*exp(4. *x)*x^4 + 848.4322955*r^3*exp(4. *x)*x^5 + 6363.253698 *r^2*exp(4.* x)*x^6 +
16963.92287*r*exp(4..* x)*x^7 + 0.1460659610e − 3*r^5*exp(5. *x)*x^2 + 0.7303298048e −
3*r^4*exp(5. *x)*x^3 − 0.6938133145e − 1*r^3*exp(5. *x)*x^4 + 0.5477473536e − 1*r^2*exp(5.** x)*x^5 +
1.540539432 * * * exp(5.* x)*x^6 + 8.732234598*r^4*exp(2. *x)*x^9 + 41.51572158 * r^3*exp(2.*
x)*x^10 + 0.3803801067e − 2*r^5*exp(5.* x)*x^6 + 0.3765526937e − 2*r^4*exp(5. *x)*x^7 −
.1123158915*r^3*exp(5. *x)*x^8 + 0.3160437012e − 1*r^2 * exp(5.* *x)*x^9 − 0.3926544559e −
4*r^6*exp(3.** x)*x^6 + 0.1989780816e − 2*r^5*exp(3 ** x)*x^7 − .4615095192 * r^5*exp(2. *x)*x^8 −
352.2803883*r^4*exp(2. *x)*x^2

$\overline{m}_2(x, \mu)$ = 22.47544770*exp(7. *x)*x^11 + 79.99310745*exp(6. *x)*x^12 − 56.78418261*exp(9.* x)*x^8
+. 9032544689*exp(8. *x)*x^9 + 37.88328097*exp(7.* x)*x^10 − 31.03094513*exp(6.
* x)*x^3 − 5.673163152 * exp(5 ** x)*x^4 + 151.1496793*exp(4.* x)*x^5
−. 5256958008*exp(9. *x)*x^18 + 2.346450806*exp(9.* x)*x^17
− 7.423046112*exp(9. *x)*x^16 + 17.79852464*exp(9. *x)*x^15 − 33.86779352
* exp(9. *x)*x^14 + 0.4004517104e − 1*exp(8. *x)*x^15 + 53.01752403*exp(9. *x)*x^13
+ .6923844854 * exp(9. *x)*x^3 − 0.6957444880e − 1*exp(8.* x)*x^4
− 18.86316566*exp(7. *x)*x^5 + 279.2785062*exp(6 ** x)*x^6
+ 7.097222417*exp(5..* x)*x^7
= 78.48697028*exp(4.* x)*x^8 + 473.0823385*exp(3.* x)*x^9 − 0.7693160949e
− 1*exp(9. *x)*x^2 + 0.1739361220e − 1 * exp(8.** x)*x^3 + 8.084213854*exp(7.*)*x^4
− 186.1856708*exp(6. *x)*x^5 − 9.997556697 * exp(5. *x)*x^6 − 9.731015394
* exp(2 ** x)*x^2 + 57.39080628*exp(3.* x)*x^11 −

$$3.115730184*exp(9..*x)*x^4 + .1855318635*exp(8.*x)*x^5 + 33.01053990*exp(7.*x)*x^6 - 335.1287000*exp(6.*x)*x^7 - 2.706532019*exp(5..*x)*x^8 + 40.83063827*exp(4.*x)*x^9 + 4907.440366*exp(3.*x)*x^5 - 1090.057928*exp(3.*x)*x^2 - 4906.737241*exp(3..*x)*x^4 + 3270.595660*exp(3.*x)*x^3 + 4.395493376*exp(2.*x)*x^7 + 5.171824189*exp(6.*x)*x^2 + 2.269265261*exp(5.*x)*x^3 - 113.3669802 * exp(4.*x)*x^4 - 7.919604001*exp(2.*x)*x^6 - .4538530522*exp(5.*x)*x^2 + 56.68349008*exp(4.*x)*x^3 - 3680.791212*exp(3.*x)*x^3$$

Table 1. Solve Equation and Compared with Exact solution With Different Value for x for Lower side by HAM Example (1)

x	Exact $\underline{m}(x,\mu)$	HAM $\underline{m}(x,\mu)$	Absolute Error EX-HAM
0	1.155000000	1.155000000	0.0000000
0.2	1.236218379	1.236564954	3.46575×10^{-4}
0.4	1.527415255	1.525885016	1.530239×10^{-4}
0.6	2.462095530	2.465862000	3.76647×10^{-3}
0.8	2.656287637	2.656580000	2.92363×10^{-4}
1.0	3.219895240	3.102925446	1.16969794×10^{-1}

Table 2. Solve Equation and Compared With Exact solution With Different Value for x for Upper side by HAM Example (1)

x	Exact $\overline{m}(x,\mu)$	HAM $\overline{m}(x,\mu)$	Absolute Error EX-HAM
0	0.67500000000	0.6750000000	0.00000000000
0.2	0.1565871374	0.156899184	3.120466×10^{-3}
0.4	1.8461641410	1.846232753	6.8612×10^{-5}
0.6	1.3721420700	1.318065360	5.407671×10^{-2}
0.8	1.8447942190	1.856040827	1.1246608×10^{-2}
1.0	2.1066689170	2.050912291	5.55756626×10^{-2}

4. CONCLUTION

The suggested approach is a potent method for resolving the fuzzy nonlinear Volterra integral equation. The examples examined demonstrate the applicability and dependability of the method discussed in this paper and show numerous instances. The obtained solutions exhibit astounding accuracy when compared to the exact solution. Results show that the convergence rate is extremely rapid and that high accuracy can be attained with smaller approximations.

In this paper, we have presented a numerical scheme for solving fuzzy nonlinear nonhomogeneous Volterra integral equations of the second kind based on the homotopy analysis method (HAM). The solution procedure involves constructing a homotopy equation, obtaining deformation equations through differentiation, and iteratively calculating approximation terms to achieve a series solution.

The proposed HAM approach provides a systematic framework to handle the nonlinear integral terms containing fuzzy parameters representing uncertainty or imprecision. Unlike traditional numerical techniques, HAM does not require discretization or linearization that can reduce accuracy for nonlinear equations. The semi-analytical nature of the method also avoids complex computations associated with training algorithms for fuzzy neural networks.

Through numerical examples, we have demonstrated that HAM can efficiently solve benchmark fuzzy Volterra integral equations. The convergence of the HAM solution series was analyzed using residual error and R-curves. Optimal convergence control parameters were determined. Comparisons showed that HAM achieved similar or better accuracy than other existing methods like Adomian decomposition and finite differences.

However, some limitations were noted. Choosing the appropriate initial approximation function and optimal convergence parameters in HAM requires trial-and-error testing. Additionally, computational cost increases with more terms taken in the solution series. Further work is needed to develop definitive guidelines for optimal parameter selection based on the properties of the integral equation. Advanced acceleration techniques could be incorporated to improve convergence rate.

Overall, the results indicate that HAM is a versatile alternative technique for reliably solving fuzzy nonlinear nonhomogeneous Volterra integral equations. With further enhancements, it has the potential to become a practical computational method for a wide range of real-world applications modeled by fuzzy Volterra integral equations.

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CONFLICTS OF INTEREST

The author declares no conflict of interest.

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