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Various Closed-Form Solitonic Wave Solutions of Conformable Higher-Dimensional Fokas Model in Fluids and Plasma Physics

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ABSTRACT: This work focuses on finding closed-form analytic solutions of a higher-dimensional fractional model, in conformable sense, known by the (4+1)-dimensional Fokas equation. Fractional partial differential equations (FPDEs) and systems can describe heritable real-world occurrences. However, solving such models can be difficult, especially for nonlinear problems. The homogeneous balancing method (HBM) is investigated and extended to handle the (4+1)-dimensional Fokas equation with Kerr law nonlinearity. The HBM has the ability to solve linear and nonlinear fractional problems, incorporating the concepts of some fractional calculus principles, including fractional derivative techniques. It's important to note that there isn't a single and universally applicable method to solve such equations due to their complexity. The specific form of the equation and the initial or boundary conditions influence the solution method chosen. The results obtained from the extended HBM are compared to those in the literature to prove the strategy's efficacy. This paper proposes expanding the HB technique with result analysis to solve nonlinear FPDEs, demonstrating its feasibility and efficiency.
Keywords: Fractional Calculus, Homogeneous Balance Method (HBM), (4+1) Fokas Equation, Stability, Sensitivity.

1. INTRODUCTION

There is little doubt that fractional-order models can provide more precise and adaptable explanations for systems with long-term memory effects [1]. Fractional calculus serves as a powerful mathematical tool for describing such occurrences by allowing non-integer-order derivatives and integrals. However, it is important to note that integer-order models may still be useful in many real-world applications, and the use of fractional calculus is not always necessary or appropriate [2, 3]. Additionally, although fractional calculus has a long history dating back to the 17th century, its application in engineering and research has significantly increased recently because of technological advancements and the need for more advanced modelling tools [4]. Consequently, fractional calculus remains a new and emerging subject, with ongoing investigations and the development of new techniques and applications [5, 6]. In summary, fractional calculus proves to be a valuable mathematical tool for modelling systems with long-term memory effects; however, its application is not universally required or suitable for all applications [7, 8, 9, 10]. Many real-world situations still warrant the application of classical calculus, and the application of fractional calculus necessitates careful consideration and competence.

This observation underscores the need to solve complex models in fractional calculus theory. Accurately modelling complex real-world events using this theory is only one step in the process; getting solutions to these models is also required to completely comprehend the physical and technological aspects of the problem at hand [11]. The solutions to these models can provide useful insights into the behaviour of the system under study, allowing researchers to make predictions and devise interventions to enhance the system. However, due to the intricate dynamics involved, acquiring solutions to these models can be a challenging undertaking. Researchers have worked diligently to develop analytical approaches and algorithms that can be used to solve these models, and they are continually striving to improve them. Accurate solutions to complex models are crucial for progress in a wide range of fields, including engineering, physics

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and economics [12, 13, 14]. Fan [15] initially introduced the algebraic method, which has been further enhanced by Ibrahim et al. who proposed research on obtaining traveling wave solutions from nonlinear partial differential equations and selected the Benny-Luke equation and Vakhnenko-Parkes equation as illustrations of the method. It has been emphasised that more travelling wave solutions for nonlinear partial differential equations can be effectively obtained by employing the repeated homogeneous balance method [16]. Wafaa et al. have also explored the homogeneous balancing approach (HB) method in their studies on solving nonlinear physical problems (PDE) of generalised regularised long wave equations, modified dispersive water equations and Kawahara equations [17, 18]. This approach entails identifying a nonlinear transformation that can be applied to derive the exact solution of the problem [19, 20]. Furthermore, Ibrahim et al. utilised a modified version of the HB method to address the Riccati equation. They applied this method to the Klein-Gordon equation with Kerr law non-linearity, resulting in the achievement of numerous exact traveling wave solutions involving the integration of exponential, hyperbolic, trigonometric and rational functions [21].

Therefore, this study aims to develop the HBM with the ability to solve nonlinear applications and derive solitonic waves of fractional FPDEs, namely the (4+1)-dimensional Fokas equation.

The (4+1)-dimensional Fokas equation is a generalisation of two nonlinear evolution equations, namely the Kadomtsev-Petviashvili (KP) and Davy-Stewartson (DS) equations [22]. It has several applications in quantum field theory, solid-state physics, fluid mechanics, water wave theory, ocean dynamics and many others [23, 24, 25].

The significance of the Fokas equation suggests that the complexification of time may be explored within the framework of contemporary field theories by investigating the presence of nonlinear integrable equations in a special four-dimensional setting that incorporates complex time [22]. Therefore, it is crucial to understand the exact solution of the Fokas equation, including solitary waves, among various other solutions. Recently, Lee et al. constructed several solutions to the Fokas equation, including traveling wave solutions, by applying a modified tanh-coth technique to solve the suggested equation [26]. Yang et al. studied potential symmetries and Lie point symmetries of the Fokas equation [27]. Kim et al. employed the analytical solution obtained for the Fokas equation using Hirota's bilinear method [28].

The concept of fractional calculus will be utilised to develop a new form of HBM, wherein the methods developed will be capable of addressing the challenge of obtaining analytical solutions for the (4+1)-dimensional Fokas equation under conformable definitions of differentiability [29].

This paper is structured as follows: In the 'Method Description' section, the methodology and general analysis of the tanh-coth method are specified. The 'Mathematical Application' section contains the numerical implementation of the methods for solving the fractional Fokas equation. In the 'Results and Discussion' section, the obtained results are presented and discussed. Additionally, a proof of the stability and sensitivity analysis of the exact solution is provided. Finally, the conclusions are presented in the 'Conclusion' section.

2. PRELIMINARIES

In fractional calculus, a branch of the broader field, common calculus operators of differentiation and integration are generalised to non-integer orders. This implies that principles covered by fractional calculus include non-integer order differentiation and integration. By gaining an understanding of some basic terms and ideas connected to fractional calculus theory, as described in this section, we can enhance our comprehension of the research presented in the subsequent sections.

Definition 1 [30]: Let $f\ :[0,\infty)\to R$ be a function. Then conformable fractional derivative of f of order α is defined by:

$$T_\alpha f(x)=\lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon x^{1-\alpha}\right)-f(x)}{\varepsilon} \tag{1}$$

Where $\alpha\in(0,1)$ and it holds for all $x>0$.

If the function f is $\alpha-$differentiable in $(0,l)$ for $l>0$ and further $\lim_{x\to 0^+} f^{(\alpha)}(x)$ exists, then the conformable derivative at 0 is defined

$$f^{(\alpha)}(0)=\lim_{x\to 0^+} f^{(\alpha)}(x) \tag{2}$$

Also, the conformable integral of function f is defined as:

$$I_\alpha^l f(x)=\int_l^x \frac{f(t)}{t^{1-\alpha}}\,dt,\ l\geq 0 \text{ and } \alpha\in(0,1) \tag{3}$$

Theorem 1 [30]**:** Suppose the functions u and v are $\alpha-$differentiable at any point $x>0$ for $\alpha\in(0,1)$. We then have the following properties:

1) $T_\alpha(au+bv)=aT_\alpha(u)+bT_\alpha(v),\ a,b\in R$
2) $T_\alpha(x^n)=nx^{n-\alpha},\ n\in R$
3) $T_\alpha(k)=0,$ for any constant k
4) $T_\alpha(uv)=uT_\alpha(v)+vT_\alpha(u)$
5) $T_\alpha\left(\frac{u}{v}\right)=\frac{vT_\alpha(u)-uT_\alpha(v)}{v^2},\ a,b\in R$
6) Additionally,if the function v is differentiable, then $T_\alpha(v)_{(x)}=x^{l-\alpha}\frac{dv}{dx}$

3. DESCRIPTION OF METHOD AND ANALYSIS

Consider a nonlinear conformable fractional equation to explain the fundamental concept behind our approach,

$$F\left(u,u_x,u_\tau,D_\tau^\alpha u,D_x^\alpha u,D_\tau^{2\alpha}u,D_x^{2\alpha}u,\ldots\right)=0,\ 0<\alpha<1 \tag{4}$$

Where F polynomial of a function and its partial fractional derivatives. The steps of the proposed technique are as follows:

Step 1: Assume the complex transformation operator is considered as follows:

$$\Theta(x,t)=U(\xi)e^{ir(x,t)},r(x,t)=\eta x-zt,\xi=A\frac{x^\alpha}{\alpha}+B\frac{y^\alpha}{\alpha}+C\frac{z^\alpha}{\alpha}+\psi\frac{w^\alpha}{\alpha}-\chi\frac{t^\alpha}{\alpha} \tag{5}$$

Where z and η are unknown nonzero constants. The nonlinear FPDE in Eq. (4) is transformed into the non-linear ordinary differential equation:

$$H\left(U,U',U'',U''',\ldots\right)=0 \tag{6}$$

Here, H is a function of $U(\xi)$, and the prime indicates its derivatives with respect to r

Step 2: Suppose that Eq. (7) has a solution of the form:

$$U=a_0+\sum_{i=1}^{m}\left(a_i\varphi^i+b_i\varphi^{-i}\right) \tag{7}$$

Where $a_i,b_i,\ i=0,1,2,\ldots n$ are constants to be computed later, n is a positive integer chosen by balancing the highest order derivatives term with the nonlinear term in Eq. (2), and φ satisfy the following equation [31]

$$\varphi'=\ln(\rho)\left(\mu+v\varphi+\lambda\varphi^2\right) \tag{8}$$

Where μ,v and λ are real numbers.

Step 3: Eq. (7) may be substituted into Eq. (8), yielding an algebraic system of equations The solution to this system can be obtained using Maple, and the solutions to Eq. (11) are all precise solutions to Eq. (9).

4. APPLICATION OF THE HB METHOD

In this section, we explore the space-time conformable fractional order (4+1)-dimensional Fokas equation with the help of the modified homogeneous balance method.

$$4\frac{\partial^{2\alpha}u}{\partial t^\alpha\partial x^\alpha}-\frac{\partial^{3\alpha}}{\partial x^{3\alpha}}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^3 u}{\partial y^3}\right)+12\frac{\partial^\alpha u}{\partial x^\alpha}\left(\frac{\partial u}{\partial y}\right)+12u\frac{\partial^\alpha u}{\partial x^\alpha}\left(\frac{\partial u}{\partial y}\right)-6\frac{\partial^2 u}{\partial z\,\partial w}=0,\ 0<\alpha<1 \tag{9}$$

Where u is an unknown function of time t and 4 spatial dimensions x, y, z, w. This expanded model, Eq. (9), is anticipated to elucidate wave phenomena in complex medium. It may capture the dispersion or distortion of surface waves, offering insights into various phenomena associated with the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations.

Furthermore, considering the aforementioned facts concerning fractional derivatives and the importance of the Fokas equation, it becomes crucial to investigate the new soliton solutions of Eq. (9). Now, let's consider the following traveling wave transformation, which transforms the independent variables into a single variable:

$$\xi = A\frac{x^{\alpha}}{\alpha} + B\frac{y^{\alpha}}{\alpha} + C\frac{z^{\alpha}}{\alpha} + \psi\frac{w^{\alpha}}{\alpha} - \chi\frac{t^{\alpha}}{\alpha} \tag{10}$$

Where A, B, C, ψ, χ are real constants. The variable x is explained as the propagation distance, and t is the slow time. By using the transformations of Eq. (6), then Eq. (5) is then converted into:

$$AB\left(A^2 - B^2\right)U'' + \left(4A\psi + 6C\chi\right)U - 6ABU^2 = 0 \tag{11}$$

Operating the complex transformation

$$\Theta(t,x) = U(x,t)e^{ir(x,t)}, \; r(x,t) = \eta x - zt \tag{12}$$

Where z and η are the constants. Balancing u'' with u^2 in Eq. (12) gives $m + 2 = 2m$ so that $m = 2$. Thus, the modified homogeneous balance method admits the following solution:

$$U = a_0 + a_1\varphi + a_2\varphi^2 + b_1\varphi^{-1} + b_2\varphi^{-2} \tag{13}$$

By substituting Eq. (12), Eq. (13) in Eq. (11), along with derivatives of U, and equating the coefficients of the power φ^i and equal to zero, we obtain a set of equations. By solving this system of algebraic equation with the help of the Maple software, we obtain:

Case1: $a_0 = \frac{1}{4}\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{AB}$, $a_1 = v\lambda\left(A^2 - B^2\right)\ln(\rho)^2$, $b_1 = 0$.

$a_2 = \left(A^2 - B^2\right)\lambda^2\ln(\rho)^2$, $b_2 = 0$ and $\mu = \frac{1}{4}\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{AB\lambda\left(A^2 - B^2\right)\ln(\rho)^2}$.

1) When $v^2 - 4\mu\lambda < 0$ with $\lambda \neq 0$

$$\Theta_{11} = \left[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB} + \left(A^2 - B^2\right)\ln(\rho)^2 v\left(-\frac{v}{2} + \frac{\sqrt{Ri}}{2}\tan_{\rho}\left(\frac{\sqrt{-R}}{2}\right)\xi\right)\right.$$
$$\left. + A^2\ln(\rho)^2 - B^2\ln(\rho)^2\left(-\frac{v}{2} + \frac{\sqrt{Ri}}{2}\tan_{\rho}\left(\frac{\sqrt{-R}}{2}\right)\xi\right)\right]e^{i(\eta x - zt)}$$

$$\Theta_{12} = \left[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB} + \left(A^2 - B^2\right)\ln(\rho)^2 v\left(-\frac{v}{2} + \frac{\sqrt{Ri}}{2}\cot_{\rho}\left(\frac{\sqrt{-R}}{2}\right)\xi\right)\right.$$
$$\left. + A^2\ln(\rho)^2 - B^2\ln(\rho)^2\left(-\frac{v}{2} + \frac{\sqrt{Ri}}{2}\cot_{\rho}\left(\frac{\sqrt{-R}}{2}\right)\xi\right)\right]e^{i(\eta x - zt)}$$

$$\Theta_{13} = \left[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB}\right.$$
$$+ \left(A^2 - B^2\right)\ln(\rho)^2 v\left(-\frac{v}{2} + \frac{\sqrt{Ri}}{2}\left(\tan_{\rho}\left(\sqrt{-R}\right)\xi \pm \sqrt{mn}\sec_{\rho}\left(\sqrt{-R}\right)\xi\right)\right)$$

$$+A^2\ln(\rho)^2-B^2\ln(\rho)^2\left(-\frac{v}{2}+\frac{\sqrt{Ri}}{2}\left(\tan_\rho\left(\sqrt{-R}\right)\xi\pm\sqrt{mn}\sec_\rho\left(\sqrt{-R}\right)\xi\right)\right)\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{14}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\left(-\frac{v}{2}+\frac{\sqrt{Ri}}{2}\left(\cot_\rho\left(\sqrt{-R}\right)\xi\pm\sqrt{mn}\csc_\rho\left(\sqrt{-R}\right)\xi\right)\right)$$

$$+A^2\ln(\rho)^2-B^2\ln(\rho)^2\left(-\frac{v}{2}+\frac{\sqrt{Ri}}{2}\left(\cot_\rho\left(\sqrt{-R}\right)\xi\pm\sqrt{mn}\csc_\rho\left(\sqrt{-R}\right)\xi\right)\right)\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{15}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\left(-\frac{v}{2}+\frac{\sqrt{-R}}{2}\left(\tan_\rho\left(\sqrt{-R}\right)\xi-\sqrt{mn}\cot_\rho\left(\sqrt{-R}\right)\xi\right)\right)$$

$$+A^2\ln(\rho)^2-B^2\ln(\rho)^2\left(-\frac{v}{2}+\frac{\sqrt{-R}}{2}\left(\tan_\rho\left(\sqrt{-R}\right)\xi\pm\sqrt{mn}\cot_\rho\left(\sqrt{-R}\right)\xi\right)\right)\Bigg]e^{i(\eta x-zt)}$$

2) When $R>0$ with $\lambda\neq 0$

$$\Theta_{16}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}+\left(A^2-B^2\right)\ln(\rho)^2v\left(-\frac{v}{2}+\frac{\sqrt{R}}{2}\left(\tanh_\rho\left(\sqrt{\frac{R}{2}}\right)\xi\right)\right)$$

$$+A^2\ln(\rho)^2-B^2\ln(\rho)^2\left(-\frac{v}{2}+\frac{\sqrt{R}}{2}\left(\tanh_\rho\left(\frac{\sqrt{R}}{2}\right)\xi\right)\right)\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{17}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}+\left(A^2-B^2\right)\ln(\rho)^2v\left(-\frac{v}{2}+\frac{\sqrt{R}}{2}\left(\coth_\rho\left(\sqrt{\frac{R}{2}}\right)\xi\right)\right)$$

$$+A^2\ln(\rho)^2-B^2\ln(\rho)^2\left(-\frac{v}{2}+\frac{\sqrt{R}}{2}\left(\coth_\rho\left(\frac{\sqrt{R}}{2}\right)\xi\right)\right)\Bigg]e^{i(\eta x-zt)}$$

3) When $\mu\lambda>0$ with $v=1$

$$\Theta_{18}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\,\lambda\Bigg[\frac{1}{2}\xi\sqrt{\frac{\mu}{\lambda}}\tan_\rho\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\tan_\rho\left(\frac{1}{2}\xi\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}\right)^2\right)\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{19}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\lambda\left[\frac{1}{2}\xi\sqrt{\frac{\mu}{\lambda}}\cot_\rho\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}\right.$$

$$+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\cot_\rho\left(\frac{1}{2}\xi\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}\right)^2\right)\Bigg]e^{i(\eta x-zt)}$$

4) When $\mu\lambda<0$ with $v\neq 1$

$$\Theta_{110}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\lambda\left[-\sqrt{\frac{\mu}{\lambda}}\tanh_\rho\left(\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}\ i\right)\xi\right.$$

$$+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(-\sqrt{\frac{\mu}{\lambda}}i\ \tanh_\rho\left(\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}\ i\xi\right)^2\right)\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{111}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\lambda\left[-\sqrt{\frac{\mu}{\lambda}}i\ \coth_\rho\left(\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}\ i\right)\xi\right.$$

$$+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(-\sqrt{\frac{\mu}{\lambda}}i\ \coth_\rho\left(\sqrt{\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB\left(A^2-B^2\right)\ln(\rho)^2}}\ i\xi\right)^2\right)\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{112}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\left[-\frac{1}{2}i\sqrt{\frac{\mu}{\lambda}}\left(\tanh_\rho\left(\frac{1}{2}\sqrt{-\mu\lambda}i\right)\xi-\coth_\rho\left(\frac{1}{2}\sqrt{-\mu\lambda}i\right)\xi\right)\right.$$

$$+\left(A^{2}-B^{2}\right)\ln(\rho)^{2}\lambda^{2}\left[-\frac{1}{2}i\sqrt{\frac{\mu}{\lambda}}\left(\tanh_{\rho}\left(\frac{1}{2}\sqrt{-\mu\lambda i}\right)\xi-\coth_{\rho}\left(\frac{1}{2}\sqrt{-\mu\lambda i}\right)\xi\right)\right]e^{i(\eta x-zt)}$$

5) When $v=0$ with $\mu=\lambda$

$$\Theta_{113}=\left[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}\right.$$

$$+\left(A^{2}-B^{2}\right)\ln(\rho)^{2}v\lambda\left(\tan_{\rho}\left(\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\right)\xi\right)$$

$$+\left(A^{2}-B^{2}\right)\ln(\rho)^{2}\lambda^{2}\left(\tan_{\rho}\left(\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\right)\xi\right)$$

$$\Theta_{114}=\left[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}\right.$$

$$+\left(A^{2}-B^{2}\right)\ln(\rho)^{2}v\lambda\left(-\cot_{\rho}\left(\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\right)\xi\right)$$

$$+\left(A^{2}-B^{2}\right)\ln(\rho)^{2}\lambda^{2}\left(-\cot_{\rho}\left(\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\right)\xi\right)$$

$$\Theta_{115}=\left[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}+\frac{1}{2}\left(A^{2}-B^{2}\right)\ln(\rho)^{2}v\lambda\left(\tan_{\rho}\left(\frac{\mu}{2}\right)\xi-\cot_{\rho}\left(\frac{\mu}{2}\right)\xi\right)\right.$$

$$\left.+\frac{1}{4}\left(A^{2}-B^{2}\right)\ln(\rho)^{2}\lambda^{2}\left(\tan_{\rho}\left(\frac{\mu}{2}\right)\xi-\cot_{\rho}\left(\frac{\mu}{2}\right)\xi\right)\right]e^{i(\eta x-zt)}$$

6) When $v=0$ with $\mu=-\lambda$

$$\Theta_{116}=\left[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}\right.$$

$$-\left(A^{2}-B^{2}\right)\ln(\rho)^{2}v\lambda\left(\tanh_{\rho}\left(\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\right)\xi\right)$$

$$\left.+\left(A^{2}-B^{2}\right)\ln(\rho)^{2}\lambda^{2}\left(\tanh_{\rho}\left(\left(\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\right)\xi\right)^{2}\right)\right]e^{i(\eta x-zt)}$$

$$\Theta_{117}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\lambda\left(-\coth_\rho\left(\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)$$

$$\left.+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(-\coth_\rho\left(\left(\left(\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)^2\right)\right)\right]e^{i(\eta x-zt)}$$

$$\Theta_{118}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\lambda\left(-\tanh_\rho(2\mu)\xi\pm i\sqrt{mn}\ \mathrm{sech}_\rho(2\mu)\xi\right)$$

$$\left.+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(\left(-\tanh_\rho(2\mu)\xi\pm i\sqrt{mn}\ \mathrm{sech}_\rho(2\mu)\xi\right)^2\right)\right]e^{i(\eta x-zt)}$$

$$\Theta_{119}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\lambda\left(-\coth_\rho(2\mu)\xi\pm i\sqrt{mn}\ \mathrm{csch}_\rho(2\mu)\xi\right)$$

$$\left.+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(\left(-\coth_\rho(2\mu)\xi\pm i\sqrt{mn}\ \mathrm{csch}_\rho(2\mu)\xi\right)^2\right)\right]e^{i(\eta x-zt)}$$

$$\Theta_{120}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\left(A^2-B^2\right)\ln(\rho)^2v\lambda\left(-\frac{1}{2}\tanh_\rho\left(\frac{\mu}{2}\right)\xi\pm\coth_\rho\left(\frac{\mu}{2}\right)\xi\right)$$

$$\left.+\left(A^2-B^2\right)\ln(\rho)^2\lambda^2\left(\left(-\frac{1}{2}\tanh_\rho\left(\frac{\mu}{2}\right)\xi\pm\coth_\rho\left(\frac{\mu}{2}\right)\xi\right)^2\right)\right]e^{i(\eta x-zt)}$$

Case 2: $a_0=\frac{1}{4}\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{AB}$, $a_1=0,\ a_2=0$

$b_1=\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4AB\lambda}$, $b_2=\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16A^2B^2\lambda^2\left(A^2-B^2\right)\ln(\rho)^2}$

and $\mu=\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB\lambda\left(A^2-B^2\right)\ln(\rho)^2}$.

1) When $R = v^2 - 4\mu\lambda < 0$ with $\lambda \neq 0$

$$\Theta_{21} = \Bigg[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)v}{4AB}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\tan_p\left(\frac{\sqrt{-R}\,i}{2}\right)\xi\right) + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\tan_p\left(\frac{\sqrt{-R}\,i}{2}\right)\xi\right)\Bigg] e^{i(\eta x - zt)}$$

$$\Theta_{22} = \Bigg[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)v}{4AB}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\cot_p\left(\frac{\sqrt{-R}\,i}{2}\right)\xi\right)^{-1} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\cot_p\left(\frac{\sqrt{-R}\,i}{2}\right)\xi\right)^{-2}\Bigg] e^{i(\eta x - zt)}$$

$$\Theta_{23} = \Bigg[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)v}{4AB}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\left(\tan_p\left(\sqrt{-R}\right)\xi \pm \sqrt{mn}\sec_\rho\left(\sqrt{-R}\right)\xi\right)\right)^{-1} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\left(\tan_p\left(\sqrt{-R}\right)\xi \pm \sqrt{mn}\sec_\rho\left(\sqrt{-R}\right)\xi\right)\right)^{-2}\Bigg] e^{i(\eta x - zt)}$$

$$\Theta_{24} = \Bigg[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)v}{4AB}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\left(\cot_p\left(\sqrt{-R}\right)\xi \pm \sqrt{mn}\csc_\rho\left(\sqrt{-R}\right)\xi\right)\right)^{-1} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\frac{v}{2} + \frac{\sqrt{R}\,i}{2}\left(\cot_p\left(\sqrt{-R}\right)\xi \pm \sqrt{mn}\csc_\rho\left(\sqrt{-R}\right)\xi\right)\right)^{-2}\Bigg] e^{i(\eta x - zt)}$$

2) When $R > 0$ with $\lambda \neq 0$

$$\Theta_{25} = \Bigg[\frac{A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi}{4AB} + \frac{\left(A^3B\ln(\rho)^2v^2 - AB^3\ln(\rho)^2v^2 + 4A\psi - 6c\chi\right)v}{4AB}\left(-\frac{v}{2} - \frac{\sqrt{R}\,i}{2}\tanh_p\left(\frac{\sqrt{R}\,i}{2}\right)\xi\right)$$

$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)^{2}}{16A^{2}B^{2}\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\left(-\frac{v}{2}-\frac{\sqrt{R}i}{2}\tanh_{p}\left(\frac{\sqrt{R}i}{2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{26}=\Bigg[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)v}{4AB}\left(-\frac{v}{2}-\frac{\sqrt{R}i}{2}\coth_{p}\left(\frac{\sqrt{R}i}{2}\right)\xi\right)$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)^{2}}{16A^{2}B^{2}\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\left(-\frac{v}{2}-\frac{\sqrt{R}i}{2}\coth_{p}\left(\frac{\sqrt{R}i}{2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{27}=\Bigg[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)v}{4AB}\left(-\frac{v}{2}-\frac{\sqrt{R}i}{2}\coth_{p}\left(\frac{\sqrt{R}i}{2}\right)\xi\right)$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)^{2}}{16A^{2}B^{2}\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\left(-\frac{v}{2}-\frac{\sqrt{R}i}{2}\coth_{p}\left(\frac{\sqrt{R}i}{2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

3) When $\mu\lambda>0$ with $v=1$

$$\Theta_{28}=\Bigg[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)v}{4AB}\left(\frac{1}{2}\frac{\sqrt{\mu}}{\lambda}\tan_{p}\left(\frac{\sqrt{\mu\lambda}}{2}\right)\xi\right)$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)^{2}}{16A^{2}B^{2}\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\left(\frac{1}{2}\frac{\sqrt{\mu}}{\lambda}\tan_{p}\left(\frac{\sqrt{\mu\lambda}}{2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{29}=\Bigg[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)v}{4AB}\left(-\frac{\sqrt{\mu}}{\lambda}i\tanh_{p}\left(\sqrt{\mu\lambda}i\right)\xi\right)$$
$$+\frac{\left(A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi\right)^{2}}{16A^{2}B^{2}\left(A^{2}-B^{2}\right)\ln(\rho)^{2}}\left(-\frac{\sqrt{\mu}}{\lambda}i\tanh_{p}\left(\sqrt{\mu\lambda}i\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{210}=\Bigg[\frac{A^{3}B\ln(\rho)^{2}v^{2}-AB^{3}\ln(\rho)^{2}v^{2}+4A\psi-6c\chi}{4AB}$$

$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4AB}\left(-\frac{\sqrt{\mu}}{\lambda}i\ \tanh_p\left(\sqrt{\mu\lambda}i\right)\xi\right)^{-1}$$

$$\left.+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\frac{\sqrt{\mu}}{\lambda}i\ \tanh_p\left(\sqrt{\mu\lambda}i\right)\xi\right)^{-2}\right]e^{i(\eta x-zt)}$$

4) When $\mu\lambda<0$ with $v\neq1$

$$\Theta_{211}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4AB}\left(-\frac{\sqrt{\mu}}{\lambda}i\ \coth_p\left(\sqrt{\mu\lambda}i\right)\xi\right)^{-1}$$

$$\left.+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\frac{\sqrt{\mu}}{\lambda}i\ \coth_p\left(\sqrt{\mu\lambda}i\right)\xi\right)^{-2}\right]e^{i(\eta x-zt)}$$

$$\Theta_{212}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4AB}\left(-\frac{1}{2}i\sqrt{\frac{\mu}{\lambda}}\left(\tanh_p\left(\frac{\sqrt{\mu\lambda}}{2}i\right)\xi+\coth_p\left(\frac{\sqrt{\mu\lambda}}{2}i\right)\xi\right)\right)^{-1}$$

$$\left.+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(i\sqrt{\frac{\mu}{\lambda}}\left(\tanh_p\left(\frac{\sqrt{\mu\lambda}}{2}i\right)\xi+\coth_p\left(\frac{\sqrt{\mu\lambda}}{2}i\right)\xi\right)\right)^{-2}\right]e^{i(\eta x-zt)}$$

5) When $v=0$ with $\mu=\lambda$

$$\Theta_{213}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4AB}\left(\tan_\rho\left(\frac{\gamma^2v^2\ln(\rho)^2+\gamma^2}{\gamma^2\lambda\ln(\rho)^2}\right)\xi\right)^{-1}$$

$$\left.+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(\tan_\rho\left(\frac{\gamma^2v^2\ln(\rho)^2+\gamma^2}{\gamma^2\lambda\ln(\rho)^2}\right)\xi\right)^{-2}\right]e^{i(\eta x-zt)}$$

$$\Theta_{214}=\left[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}\right.$$

$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4\lambda AB}\left(-\cot_\rho\left(\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{4AB\lambda\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)^{-1}$$

$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16\lambda^2A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\cot_\rho\left(\left(\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{4AB\lambda\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{215}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4\lambda AB}\left(\tan_\rho\left(\frac{\mu}{2}\right)\xi-\cot_\rho\left(\frac{\mu}{2}\right)\xi\right)^{-1}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16\lambda^2A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(\tan_\rho\left(\frac{\mu}{2}\right)\xi-\cot_\rho\left(\frac{\mu}{2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

6) When $v=0$ with $\mu=-\lambda$

$$\Theta_{216}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4\lambda AB}\left(\tanh_\rho\left(\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB\lambda\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)^{-1}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16\lambda^2A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(\tanh_\rho\left(\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB\lambda\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{217}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4\lambda AB}\left(-\coth_\rho\left(\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB\lambda\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)^{-1}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16\lambda^2A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\coth_\rho\left(\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB\lambda\left(A^2-B^2\right)\ln(\rho)^2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

$$\Theta_{218}=\Bigg[\frac{A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi}{4AB}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)v}{4\lambda AB}\left(-\tanh_\rho(2\mu)\xi\pm i\sqrt{nm}\,\operatorname{sech}_\rho(2\mu)\xi\right)^{-1}$$
$$+\frac{\left(A^3B\ln(\rho)^2v^2-AB^3\ln(\rho)^2v^2+4A\psi-6c\chi\right)^2}{16\lambda^2A^2B^2\left(A^2-B^2\right)\ln(\rho)^2}\left(-\tanh_\rho(2\mu)\xi\pm i\sqrt{nm}\,\operatorname{sech}_\rho(2\mu)\xi\right)^{-2}\Bigg]e^{i(\eta x-zt)}$$

Now, we will compare our method in equation θ_{11}, θ_{12}, θ_{16}, if $\psi=v=z=\eta=0$, $\ln(\rho)=1$, $c=4$, $A=1$, $B=0.5$, $R=-4$, $\chi=\frac{1}{16}$ with the exact solution of equation (4.11), (4.12) solved by $F-$expansion methods under the value $A_0 = -0.75$, $B_1 = 1$, $B_2 = 0.5$, $h_0 = 0.5$, $h_2 = 2$, $\xi_0 = 0$ and the same method under $\alpha_1 = 1$, $\alpha_2 = 0.5$, $d_0 = 0.5$, $d_1 = 2$, $\xi_0 = 0$ of equation u_{21} and u_{22} we obtained the same solution shown in Table 1, as follows.

Table1. Comparison between exact solution of HB method with a different method [23, 24, 25]

The F – expansion method	The proposed method
-If $A_0 = -\frac{3}{4}$, $B_1 = 1$, $B_2 = \frac{1}{2}$, $h_0 = \frac{1}{2}$, $h_2 = 2$, $\xi_0 = 0$ then $u_{21} = -\frac{3}{4} - \frac{3}{4}\tan^2(\xi)$ and $u_{22} = -\frac{3}{4} - \frac{3}{4}\tanh^2(i\xi)$. -In another approach of a same method $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, $d_0 = \frac{1}{2}$, $d_1 = 2$, $\xi_0 = 0$ then $u_{21} = -\frac{3}{4}sec^2(\xi)$ and $u_{22} = -\frac{3}{4} - \frac{3}{4}\tanh^2(i\xi)$.	- If $\psi = v = z = \eta = 0$, $\ln(\rho) = 1$, $c = 4$, $A = 1$, $B = \frac{1}{2}$, $R = -4$, $\chi = \frac{1}{16}$ then $\theta_{11} = -\frac{3}{4}\sec^2(\xi)$ and $\theta_{16} = -\frac{3}{4} - \frac{3}{4}\tanh^2(i\xi)$.

5. STABILITY ANALYSIS

This section discusses the stability of soliton solutions as well as the properties of the Hamiltonian system (HSM). The momentum of the HSM is given by [32, 33]:

$$M = 0.5\int_{a_1}^{a_2} U^2 d\xi \tag{14}$$

Where U is the independent variable and a_1, a_2 are arbitrary constants. The stability of the obtained solutions is determined by the HSM when the following criterion is $\frac{\partial M}{\partial \chi} > 0$ where χ is the speed of waves.

The solutions presented meet the stability criterion for the parameter values indicated. For example, the solutions to the space-time conformable fractional order (4+1)-dimensional Fokas equation in Eq. (6) are 0.0073792 of Eq. (11) if $t = 0.5$, $\chi = 5$ and the stability of Eq. (12) on the $t = 0.5$ and $\chi = -0.5$ are 17.30719315, which satisfy the stability condition for the indicated parameter values. As a result, given the various values of the parameters in the supplied range, the soliton solutions of the other families also satisfy the requirement, which is graphically represented in the next section.

6. SENSITIVITY ANALYSIS

In order to analyse the sensitivity of the considered model in Eq. (9), applying the Galilean operator [34] in Eq.(9) results the following dynamical system:

$$U' = V\ ;\ V' = \frac{2\left(3ABU^2 - 3cU\chi - 2AU\psi\right)}{AB\left(A^2 - B^2\right)}$$

The dynamical system is considered to have low sensitivity if a small change in starting values results in only a slight modification in the system dynamics. The critical points of the given system will be determined as follows:

$$(0,0);\ \left(0, \frac{3c\chi + 2A\psi}{3AB}\right).$$

Based on the numerical values: $A = 0.7$; $B = 1.4$; $c = 0.6$; $\chi = 0.8$; $\psi = 1.5$ the initial datum of the system is considered to be closed to the origin as follows: $P_1 = (0.04, 0)$; $P_2 = (0.05, 0)$ and $P_3 = (0.06, 0)$ the solutions are illustrated in Fig. 1(a). Subject to the given parameters, the critical points are the origin and $(1.2, 0)$. The influence, with $P_1 = (1.25, 0)$; $P_2 = (1.275, 0)$ and $P_3 = (1.3, 0)$ is also considered in Fig. 1(b).

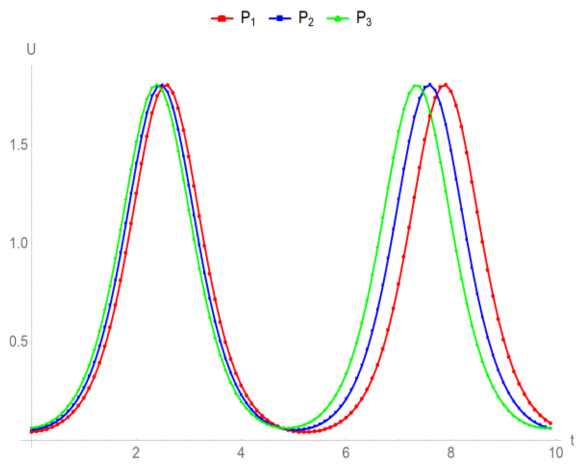

Figure 1. (a) The sensitivity assessment of for different initial values

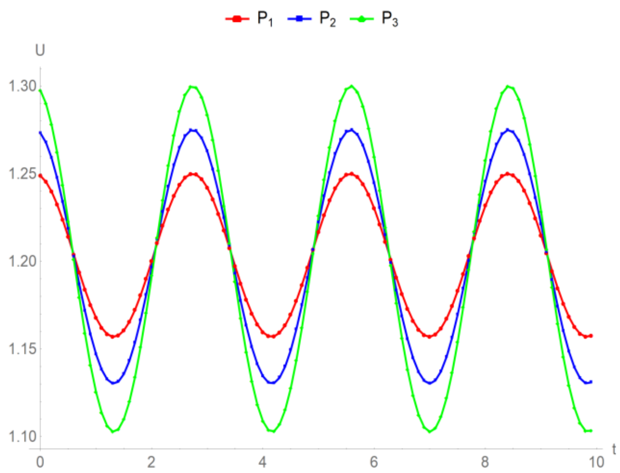

Figure 1. (b) The sensitivity assessment of for different initial values

6. RESULT AND DISCUSSION

To obtain the soliton solution, the (4+1)-dimensional Fokas equation is solved using the efficient homogeneous technique. The development of new and more comprehensive solutions for fractional order at various parameter values is the main goal of this research. The literature has developed a variety of solutions using various methods, including trigonometric, hyperbolic, and rational forms of solitary wave solutions using the HB approach. We discovered that our strategy is fresh and more thorough when compared to earlier findings. A novel simulation of 3D graphs called a contour graph gives more specific information about the physical characteristics of the precise solution. The physical behavior of Eq. (9) is summarized and illustrated in Figures 2-7.

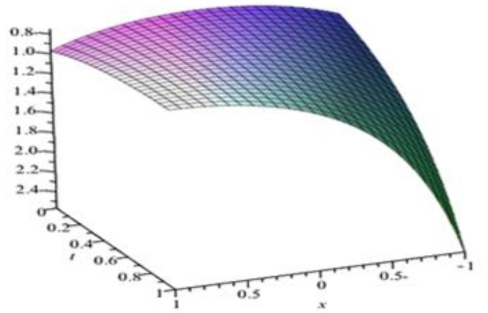

Figure 2. 3D graphics of θ_{12} for $\psi = v = z = \eta = 0$, $\ln(\rho) = 1, c = 4, \quad A = 0.5, B = \frac{1}{2}, R = -4,.$ $\chi = \frac{1}{16}, x = -1,\dots,1$ and $t = 0,\dots,1..$

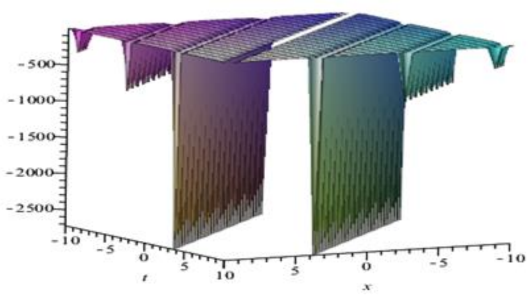

Figure 3. 3D graphics of θ_{11} for $\psi = v = z = \eta = 0$, $\ln(\rho) = 1, c = 4, \quad A = 0.5, B = \frac{1}{2}, R = -4,$ $\chi = \frac{1}{16}, x = -10,\dots,10$ and $t = -10,\dots,10.$

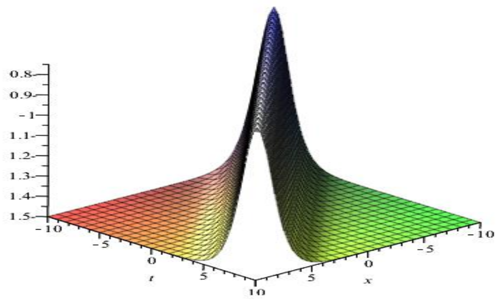

Figure 4. 3D graphics of θ_{19} for $\psi = v = z = \eta = 0$, $\ln(\rho) = 1, c = 0$, $A = 0.5, B = 0, R = -4$, $\chi = 0.5, x = -10, ..., 10$ and $t = -10, ...10$

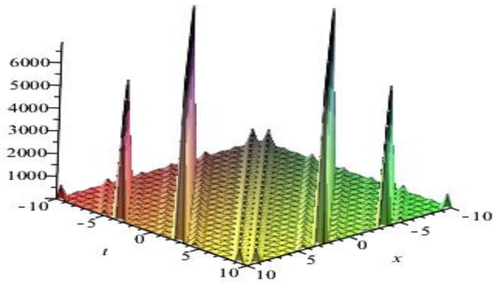

Figure 5. 3D graphics of θ_{16} for $\psi = v = z = \eta = 0$, $\ln(\rho) = 1, c = 0$, $A = 0.5, B = 0, R = -4$, $\chi = 0.5, x = -10, ..., 10$ and $t = -10, ...10$

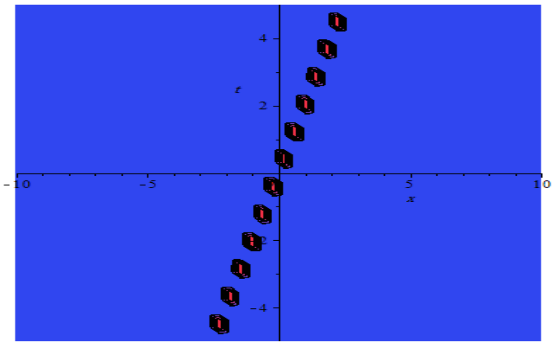

Figure 6. Contour plot of θ_{16} for $\mu = 1, \lambda = -1, \ln(\rho) = 1$, $v = 0, \chi = 0.5, x = -5, \ldots, 5$ and $t = -5, \ldots, 5$ and contour=3, filledregion=true.

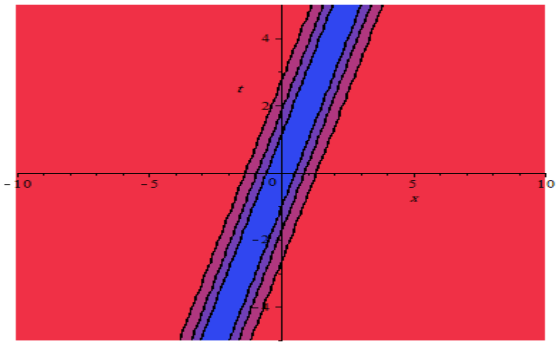

Figure 7. Contour plot of θ_{17} for $\mu = 1, \lambda = -1, \ln(\rho) = 1$, $\ln(\rho) = 1, v = 0, \chi = 1, x = -10, \ldots, 10$ and contour=3, filledregion=true.

According to Figures 2-7, the type wave is described from the solution of $\theta_{11}(x,t)$ demonstrated in Fig. 2 for choosing the parameter value for $\psi = v = z = \eta = 0$, $\ln(\rho) = 1$, $c = 4$, $A = 0.5$, $B = \frac{1}{2}$, $R = -4$, $\chi = \frac{1}{16}$, $x = -1, \ldots, 1$ and $t = 0, \ldots, 1$. Fig. 3 represented 3D of the solitary wave solution of $\theta_{12}(x,t)$ at distinct values of parameters $\psi = v = z = \eta = 0$, $\ln(\rho) = 1$, $c = 4$, $A = 0.5$, $B = \frac{1}{2}$, $R = -4$, $\chi = \frac{1}{16}$, with $x \in [-10,10]$ and $t \in [-10,10]$. Fig. 4 shows the wave structure solution for different values of $\theta_{19}(x,t)$ for $\psi = v = z = \eta = 0$, $\ln(\rho) = 1$, $c = 4$, $A = 0.5, B = 0, R = -4, \chi = \frac{1}{2}$, $x = -10, ..., 10$ and $t = -10, ..., 10$ in Fig. 5. Also in Fig. 6 contour plot of θ_{16} for $\mu = 1, \lambda = -1, \ln(\rho) = 1, v = 0, \chi = 0.5, x = -5, \ldots, 5$ and $t = -5, \ldots, 5$ Contour=3, filledregion=true, and contour plot of θ_{17} for $\mu = 1, \lambda = -1, \ln(\rho) = 1, v = 0, \chi = 1, x = -10, \ldots, 10$ and $t = -5, \ldots, 5$ Cntour=3, filledregion=true in Fig.7.

7. CONCLUSION

With their ability to simulate and explain complicated phenomena, such as nonlinear models, FPDEs have grown significantly in significance among researchers and practitioners. The current study focused on the examination and creation of an analytical approach known as HBM for solving various FPDEs. HBM can provide close-form solutions with minimal computing labor. The HBM was utilized to explore the space-time conformable fractional order (4+1)-dimensional Fokas equation and was used as a case study to successfully test the HBM's accuracy in addressing nonlinear problems. Additionally, the stability analysis of soliton solutions utilizing the characteristics of the Hamiltonian system was provided, and it was discovered that the method provides a large selection of different and accurate soliton solutions. The plotted solutions are stable if there are no breaks or discontinuities, which is a favorable indicator. It's also worth noting that when the order of the fractional derivative decreases, the wave deviates from the center. This demonstrates that the fractional derivative has a major influence on the behavior of soliton solutions. Our focus is on identifying and studying single solutions for a certain nonlinear partial differential equation. Solitons are one-of-a-kind solutions that can maintain their shape and speed while traveling over a medium. The figures show the kink, periodic, and singular periodic, dark and bright dynamical behaviors of some obtained solutions. They are essential in the area of fluid dynamics, and plasma physics.

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CONFLICTS OF INTEREST

The author declares no conflict of interest.

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