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# The Gompertz Nadarajah-Haghighi (GoNH) Distribution Properties with Application to Real Data

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**ABSTRACT:** In this study, we propose a continuous statistical distribution consisting of four parameters based on the Gompertz family called the Gompertz Nadarajah-Haghighi (GoNH) distribution. Adding parameters to the basic distribution provides the distribution with flexibility and efficiency in analysing real-world data. The model that was recently suggested has many mathematical and statistical properties. Explicit formulas for its moments, moment-generating function, survival function, risk function, characteristic function, quantile function, expansion of pdf, and ordered statistics are only a few of the many mathematical and statistical features of the recently proposed model. The maximum likelihood estimates (MLE) method was used to estimate the model's parameters. We conducted various simulation experiments to thoroughly evaluate the small sample of MLEs. The research examined the estimators' bias and mean square error, yielding positive outcomes. The study's results demonstrated that the GoNH distribution fit better than other distributions in two real-world data applications.

**Keywords:** Nadarajah-Haghighi distribution, Moments, Quantile function, Order statistics and MLE.

## 1. Introduction

Distributions are used in many fields, such as engineering, science, geography, finance and insurance. Many fields use these distributions in lifetime analysis, finance and insurance, requiring recommendations. New attempts have expanded well-known distributions and improved data adoption services.

The Nadarajah–Haghighi distribution is an extension of the exponential distribution [1]. There is a diminishing pdf in the Nadarajah-Haghighi model. The danger rates might be rising, falling or staying the same. It cannot, however, simulate unimodal hazard rates. Numerous reliability engineering issues may involve this feature and characteristic. Selim, in 2018, presented Bayes estimation and maximum likelihood estimation (MLE) for the two Nadarajah-Haghighi distributions where unknown parameters are considered in light of the recorded values [2]. Alizadeh et al. in 2019 introduced the mathematical features, characterisations and applications of the extended exponentiated Nadarajah-Haghighi model [3] and presented the Odd Nadarajah-Haghighi (ONH) family [4], a novel generator of continuous distributions. Additionally, a novel inverted Nadarajah-Haghighi and Marshall-Olkin three-parameter lifespan distribution was introduced and studied [5]. Similarly, a novel triad of parameters known as the logistic Nadarajah–Haghighi distribution based on Nadarajah–Haghighi was introduced [6]. Wu and Gui introduced a study that focuses on estimating and predicting with a specific distribution, the Nadarajah-Haghighi model, using a certain kind of censored sample [7].

In this study, a brand-new model with four parameters is introduced that could be utilised in modelling survival data and fatigue life studies. We will focus on the Gompertz family of distributions as provided by [8]. Many studies have been presented, including one that suggested the Gompertz Fréchet (GoFr) distribution [9]. The Gompertz Flexible Weibull (GoFW) distribution was studied by [10]. Four real datasets were used to show the Discrete Gompertz (DGz-G) family by [11]. Eghwerido established the shifted Gompertz-G (ShiGo-G) family, which serves as a type of generator for classical statistical distributions. It can be used to create new continuous distributions [12].

The cdf was obtained, and the pdf distribution was derived by differentiating the cdf function of the Gompertz-G family, respectively:

$$F(x,\gamma,\theta)=1-e^{\frac{\theta}{\gamma}\{1-[1-G(x)]^{-\gamma}\}} \qquad \gamma,\theta>0\,,x\geq 0 \tag{1}$$

and

$$f(x,\gamma,\theta)=\theta g(x)[1-G(x)]^{-\gamma-1}e^{\frac{\theta}{\gamma}\{1-[1-G(x)]^{-\gamma}\}} \tag{2}$$

where $\theta$ and $\gamma$ are extra shape parameters [13].

$G(x)$ and $g(x)$ are, respectively, the baseline distribution's cdf and pdf. The Nadarajah-Haghighi distribution, which serves as our baseline distribution, is derived as follows:

$$G(x,\alpha,\lambda) = 1 - e^{\{1-(1+\lambda x)^{\alpha}\}} \tag{3}$$

and

$$g(x,\alpha,\lambda) = \alpha\lambda(1+\lambda x)^{\alpha-1} e^{\{1-(1+\lambda x)^{\alpha}} \tag{4}$$

where $\alpha, \lambda$ is considered as the scale parameter [1].

The study aims to present a statistical distribution with four parameters, examine some of its characteristics and provide a practical application that demonstrates its efficiency.

The study consisted of six parts: the first part contained the introduction and objective of the study; the second part presented the proposed statistical distribution; the third part included some statistical properties of the distribution; the fourth section focused on estimating the distribution parameters; and the fifth section demonstrated a practical application using two types of real data. Finally, it was done. The most important conclusions drawn from the study are presented at the end of the study.

## 2. Gompertz Nadarajah-Haghighi distribution

Equation (5) is derived by substituting Equation (3) into Equation (2) to get the cdf of the Gompertz Nadarajah-Haghighi (GoNH) distribution. This method was used to generate new, more efficient GoNH distributions, as follows:

$$F(x)_{GoNH} = 1 - e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]} \qquad \alpha,\theta,\gamma,\lambda > 0 \;\; and \; x \geq 0 \tag{5}$$

By deriving Equation (5), we obtained the equivalent pdf, which is as follows:

$$f(x)_{GoNH} = \theta\alpha\lambda\,(1+\lambda x)^{\alpha-1}\, e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\, e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]} \tag{6}$$

$$\alpha,\theta,\gamma,\lambda > 0 \;\; and \; x \geq 0$$

where scale parameters are denoted by θ and γ, and shape parameters are represented by λ and α.

Figures (1) and (2) depict the pdf and cdf of the GoNH distribution and the distribution of x following the NH distribution using various parameter values.

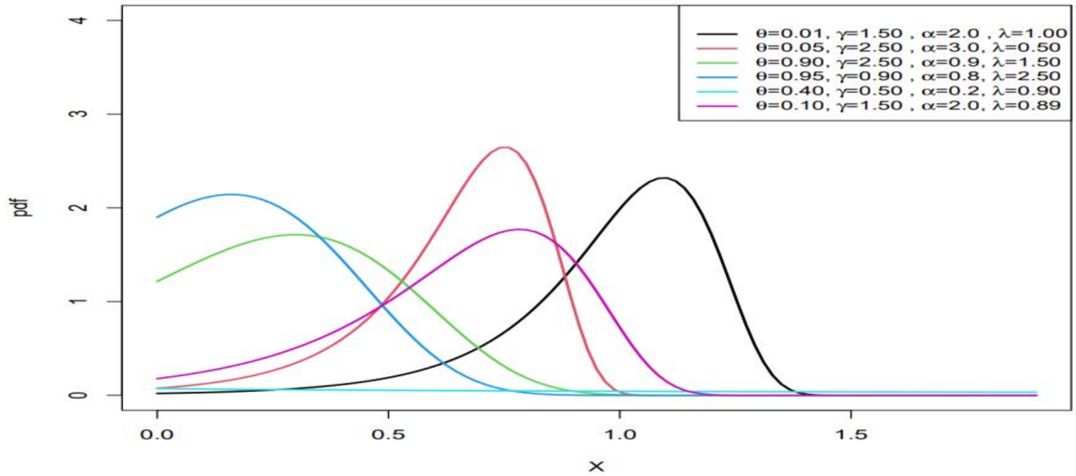


**FIGURE 1.** **pdf of GoNH distribution**

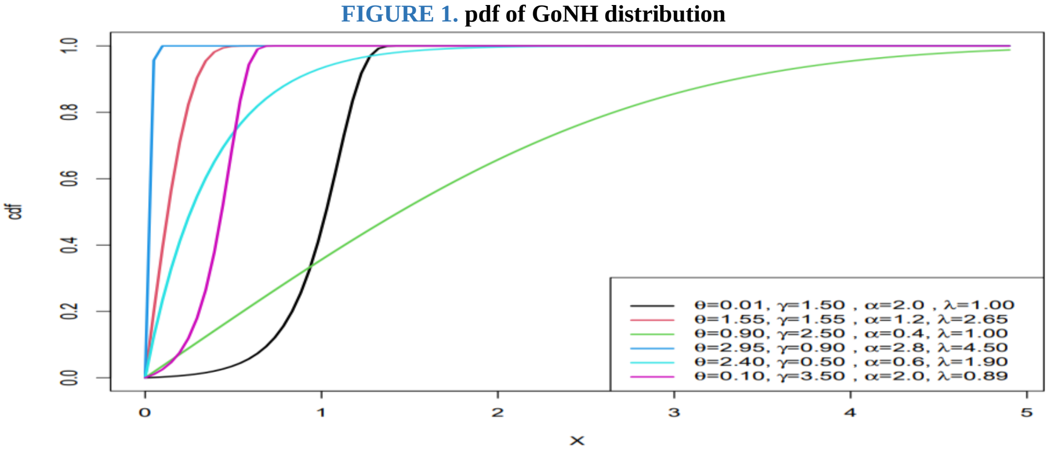


**FIGURE 2.** **cdf of GoNH distribution**

## 3. Statistical properties of the GoNH distribution

### 3.1 Various fundamental statistical characteristics of GoNH

Following is an excerpt of some of the fundamental statistical characteristics of the GoNH.
First, the relation was used to generate the dependability function: $S(x)_{GoNH} = 1 - F(x)$
Thus, the dependability function of the GoNH distribution is:

$$S(x)_{GoNH} = e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]} \tag{7}$$

In Figure (3), this is shown visually.

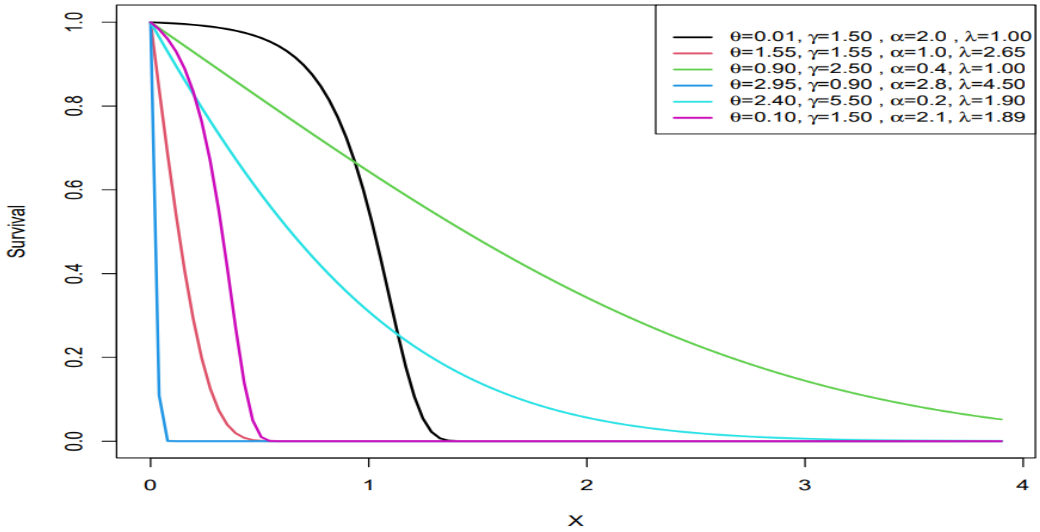


FIGURE 3. $S(x)$ of GoNH distribution

By dividing the pdf in Equation (7) and by the reliability function in Equation (8), one may obtain the failure rate of the GoNH distribution [14].

$$h(x)_{GoNH} = \frac{\theta\alpha\lambda\,(1+\lambda x)^{\alpha-1}\, e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\; e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]}}{e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]}} \tag{8}$$

Equation (8) can be rewritten as follows:

$$h(x)_{GoNH} = \theta\alpha\lambda\,(1+\lambda x)^{\alpha-1}\, e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)} \tag{9}$$

Figure (4) provides a visual representation of this.

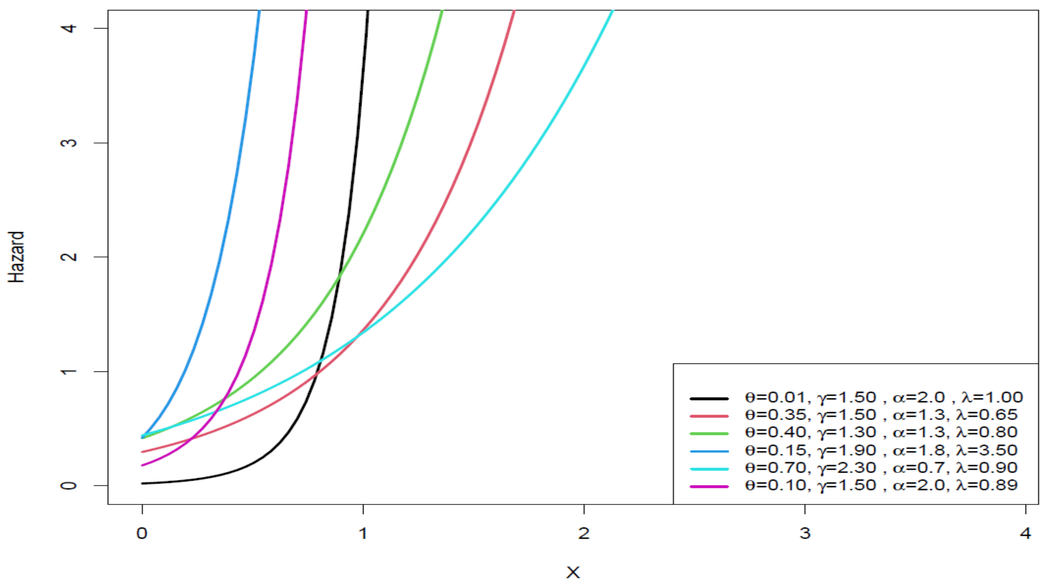


FIGURE 4. $h(x)$ of GoNH distribution

Equation (6)'s cdf is divided by Equation (8)'s reliability function to derive the odds function, which is as follows:

$$O(x)_{GoNH} = \frac{1 - e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]}}{e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]}} \tag{10}$$

Also, equations (6) and (7) are divided to provide the $r(x)$ of the GoNH distribution, which is as follows:

$$r(x)_{GoNH} = \frac{\theta\alpha\lambda\,(1+\lambda x)^{\alpha-1}\, e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\, e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]}}{1 - e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]}} \tag{11}$$

### 3.2 Quantile function

From the relationship, one may obtain the quantile function Q(u):

$$Q(u) = F^{-1}(u)$$

Thus, the GoNH distribution's quantile function can be obtained as follows:

$$Q(u)_{GoNH} = \frac{\left[1 + \frac{log\left(1 - \left[\frac{\gamma}{\theta} log(1-U)\right]\right)}{\gamma}\right]^{\frac{1}{\alpha}} - 1}{\lambda} \tag{12}$$

where U~Uniform (0,1). In other words, one may use it to produce random samples from the GoNH distribution.

$$x = \frac{\left[1 + \frac{log\left(1 - \left[\frac{\gamma}{\theta} log(1-u)\right]\right)}{\gamma}\right]^{\frac{1}{\alpha}} - 1}{\lambda}$$

It is straightforward to calculate the median of the GoNH distribution by substituting u = 0.5 into Equation (12) to obtain:

$$Median = \frac{\left[1 + \frac{log\left(1 - \left[\frac{\gamma}{\theta} log(1-(0.5))\right]\right)}{\gamma}\right]^{\frac{1}{\alpha}} - 1}{\lambda} \tag{13}$$

Also, it is possible to derive the corresponding first and third quartiles by substituting $u = 0.25$ and $u = 0.75$, sequentially, into Equation (12). Equation (12) is crucial for determining the simulation studies' parameter performance.

### 3.3 Expansions

Within this section, an expansion of the pdf will be conducted to explore the new distribution and its various properties. By taking the pdf in Equation (6) and using the series expansion and binomial expansion from [15], we can obtain:

$$f(x)_{GoNH} = \theta\alpha\lambda\,(1+\lambda x)^{\alpha-1}\, e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\, e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]}$$

By using the series expansion and $0 < e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]} < 1$, the following can be obtained:

$$f(x) = \theta\alpha\lambda\,(1+\lambda x)^{\alpha-1}\, e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)} \left[\sum_{i=0}^{\infty} \frac{\left(\frac{\theta}{\gamma}\right)^{i}\left(1-e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right)^{i}}{i!}\right]$$

By using binomial expansion, we get:

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{i}{k} \left[e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]^{k}$$

Then,

$$f(x)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\binom{i}{k}\theta\alpha\lambda\left(\frac{\theta}{\gamma}\right)^i}{i!}(1+\lambda x)^{\alpha-1}\left[e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]^{(k+1)} \tag{14}$$

Further, we can also use Equation (14) to generate expansions for many other parameters, including the moment, entropy and moment generating function (mgf).

### 3.4 Moments

The $n^{th}$ moments of the GoNH distribution are derived as follows:

$$\mu_n=E(x^n)_{GoNH}=\int_0^{\infty}x^n f(x)_{GoNH}\,dx \tag{15}$$

Then,

$$\mu_n=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\binom{i}{k}\left(\frac{\theta}{\gamma}\right)^i}{i!}\alpha\lambda\theta\int_0^{\infty}x^n(1+\lambda x)^{\alpha-1}\left[e^{-\gamma\left(1-(1+\lambda x)^{\alpha}\right)}\right]^{(k+1)}dx \tag{16}$$

By setting $t=\gamma(k+1)(1+\lambda x)^{\alpha}$ , $E(x^n)_{GoNH}$ reduces to [16]

$$\mu_n=E(x^n)_{GoNH}=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\binom{i}{k}\left(\frac{\theta}{\gamma}\right)^i}{i!}\alpha\lambda\theta e^{\gamma(k+1)}$$

$$\times\int_{\gamma(k+1)}^{\infty}\left[\left(\frac{t}{\gamma(k+1)}\right)^{\frac{1}{\alpha}}-1\right]^n\frac{e^{-t}}{\alpha\lambda^n\gamma(k+1)}dt$$

When the binomial series is applied to a power of n, the integral is:

$$\mu_n=E(x^n)_{GoNH}=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^k\binom{i}{k}\left(\frac{\theta}{\gamma}\right)^i\theta}{i!}\frac{\binom{n}{j}(-1)^{n-j}e^{\gamma(k+1)}}{\lambda^{-n}[\gamma(k+1)]^{\frac{n}{\alpha}+1}}\Gamma\left(\frac{n}{\alpha}+1,\gamma(k+1)\right) \tag{17}$$

where $\Gamma(a,n)=\int_n^{\infty}x^{a-1}e^{-x}dx$ is the gamma function with higher incomplete.

### 3.5 Moment generating function

Let X denote an R.v. The moment mgf is given by:

$$M_x(t)_{GoNH}=E(e^{tx})=\int_{-\infty}^{\infty}e^{tx}f(x)_{GoNH}\,dx \tag{18}$$

By using series expansion for $e^{tx}$, we obtain:

$$M_x(t)_{GoNH}=\sum_{n=0}^{\infty}\frac{t^n}{n!}E(x^n)=\sum_{n=0}^{\infty}\frac{t^n}{n!}[\mu_n] \tag{19}$$

Therefore, from Equation (17), the mgf of the GoNH distribution is derived as follows:

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{(-1)^k\binom{i}{k}\left(\frac{\theta}{\gamma}\right)^i\theta}{i!}\frac{\binom{n}{j}(-1)^{n-j}e^{\gamma(k+1)}}{\lambda^{-n}[\gamma(k+1)]^{\frac{n}{\alpha}+1}}\Gamma\left(\frac{n}{\alpha}+1,\gamma(k+1)\right) \tag{20}$$

### 3.6 Characteristic function

One can discover the characteristic function of the GoNH distribution by doing the following:

$$Q_x(t)_{GoNH}=E(e^{itx})=\int_0^{\infty}e^{itx}f(x)_{GoNH}\,dx \tag{21}$$

Applying the expansion of the exponential function and simplifying the equation based on the expansion, we obtain the characteristic function as follows:

$$Q_x(t)_{GoNH}=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\sum_{z=0}^{\infty}\frac{(it)^z}{z!}\frac{(-1)^k\binom{i}{k}\left(\frac{\theta}{\gamma}\right)^i\theta}{i!}\frac{\binom{n}{j}(-1)^{n-j}e^{\gamma(k+1)}\Gamma\left(\frac{n}{\alpha}+1,\gamma(k+1)\right)}{\lambda^{-n}[\gamma(k+1)]^{\frac{n}{\alpha}+1}} \tag{22}$$

### 3.7 Order statistic

For an R.s. of size n from a distribution function F(x)_GoNH, the pdf of the jth order statistic and its corresponding pdf GoNH are as follows [13]:

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} [F(x)_{GoNH}]^{j+r-1} f(x)_{GoNH} \tag{23}$$

where $F(x)_{GoNH}$ and $f(x)_{GoNH}$ stand for the pdf and cdf of the GoNH, respectively, however, the following is a R.s. of size n taken from the GoNH distribution, and the pdf of the jth order statistic is:

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} \left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma(1-(1+\lambda x)^{\alpha})}\right]}\right]^{j+r-1} \times \left[\theta\alpha\lambda\,(1+\lambda x)^{\alpha-1} e^{-\gamma(1-(1+\lambda x)^{\alpha})}\; e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma(1-(1+\lambda x)^{\alpha})}\right]}\right] \tag{24}$$

The pdf of the least order statistics can be derived by modifying Equation (24) to set j = 1.

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} \left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma(1-(1+\lambda x)^{\alpha})}\right]}\right]^{r} \times \left[\theta\alpha\lambda\,(1+\lambda x)^{\alpha-1} e^{-\gamma(1-(1+\lambda x)^{\alpha})}\; e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma(1-(1+\lambda x)^{\alpha})}\right]}\right] \tag{25}$$

However, by substituting j = n in Equation (24), the equivalent pdf of maximum order statistics can be obtained as follows:

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} \left[1 - e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma(1-(1+\lambda x)^{\alpha})}\right]}\right]^{n+r-1} \times \left[\theta\alpha\lambda\,(1+\lambda x)^{\alpha-1} e^{-\gamma(1-(1+\lambda x)^{\alpha})}\; e^{\frac{\theta}{\gamma}\left[1-e^{-\gamma(1-(1+\lambda x)^{\alpha})}\right]}\right] \tag{26}$$

## 4. Estimation

The parameters of the GoNH distribution were estimated using the MLE approach. The log-likelihood function is derived for a random sample $x_1, x_2, \ldots, x_n$ distributed according to the cdf of the GoNH distribution.

$$L(\theta, X) = (\theta\alpha\lambda)^n \prod_{i=1}^{n} \left(\sum_{i=1}^{n} (1+\lambda x_i)^{\alpha-1} \left[e^{\sum_{i=1}^{n} -\gamma(1-(1+\lambda x_i)^{\alpha})}\right]\right) \times e^{\sum_{i=1}^{n}\frac{\theta}{\gamma}\left[1-e^{\sum_{i=1}^{n} -\gamma(1-(1+\lambda x_i)^{\alpha})}\right]} \tag{27}$$

This is how one calculates the log-likelihood function L:

$$L = n\log(\theta) + n\log(\alpha) + n\log(\lambda) + (\alpha-1)\sum_{i=1}^{n} \log(1+\lambda x_i) + \sum_{i=1}^{n}(-\gamma)(1-(1+\lambda x_i)^{\alpha}) + \sum_{i=1}^{n}\frac{\theta}{\gamma}(-\gamma)(1-(1+\lambda x_i)^{\alpha}) \tag{28}$$

The way the non-linear equations $\frac{\partial \log L}{\partial \theta} = 0$, $\frac{\partial \log L}{\partial \alpha} = 0$, $\frac{\partial \log L}{\partial \lambda} = 0$ and $\frac{\partial \log L}{\partial \gamma} = 0$ are solved in [17–18] contributes to the ML parameter estimations $\alpha, \gamma, \theta, \lambda$, respectively. The only analytical way to acquire the solution was through numerical approaches using R, MAPLE and SAS programs.

## 5. Model application

Here, we demonstrate the adaptability of the GoNH distribution, which was introduced in Subsection 3.2, through two real-world applications. The first dataset below shows the frequency of failure for 50 components. For previous studies on this dataset, refer to [19–20]. The dataset is as follows: (0.036 0.058 0.061 0.074 0.078 0.086 0.102 0.103 0.114 0.116 0.148 0.183 0.192 0.254 0.262 0.379 0.381 0.538 0.570 0.574 0.590 0.618 0.645 0.961 1.228 1.600 2.006 2.054 2.804 3.058 3.076 3.147 3.625 3.704 3.931 4.073 4.393 4.534 4.893 6.274 6.816 7.896 7.904 8.022 9.337 10.940 11.020 13.880 14.730 15.080).

The second dataset contains 40 observations and has been studied in various papers, such as [21] and [22]. The dataset is as follows: (1.6 2.0 2.6 3.0 3.5 3.9 4.5 4.6 4.8 5.0 5.1 5.3 5.4 5.6 5.8 6.0 6.0 6.1 6.3 6.5 6.5 6.7 7.0 7.1 7.3 7.3 7.3 7.7 7.7 7.8 7.9 8.0 8.1 8.3 8.4 8.4 8.5 8.7 8.8 9.0).

First, when faced with unknown parameters in a variety of competing models, we estimated MLEs. Then, we conducted a comparison study using goodness-of-fit statistics, which include the Anderson-Darling (A) [23], (W) Hannan and Quinn Information Criteria (HQIC), Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC) and Consistent Akaike Information Criteria (CAIC). When these criteria were minimised, the model that performed better was identified. In order to evaluate the efficacy of the GoNH distribution, we compared it to a number of counterparts, such as the Beta Nadarajah-Haghighi (BeNH) (New), the Weibull Nadarajah-Haghighi (WeNH) (New), the [0,1]Truncated Exponentiated Exponential Nadarajah-Haghighi ([0,1]TEENH) (New), Kumaraswamy Nadarajah-Haghighi (KuNH) (New), Exponentiated Generalized Nadarajah-Haghighi (EGNH) (New), log Gamma Nadarajah-Haghighi (LGamNH) (New) and Nadarajah Haghighi [1]. Tables (2), (3) and (4), respectively, provide the MLEs and pertinent statistics for the first and second datasets.

**Table 1. Estimates of models for data 1**

| Distribution | Estimates | NLL | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|---|
| **GoNH** | $\hat{\theta} = 0.2208$ | **100.26** | **208.53** | **209.42** | **216.17** | **211.44** |
| | $\hat{\gamma} = 0.2756$ | | | | | |
| | $\hat{\alpha} = 0.3313$ | | | | | |
| | $\hat{\lambda} = 23.554$ | | | | | |
| **[0,1]TEENH** | $\hat{\theta} = 10.501$ | **102.60** | **213.21** | **214.10** | **220.86** | **216.12** |
| | $\hat{\gamma} = 0.5468$ | | | | | |
| | $\hat{\alpha} = 1.4593$ | | | | | |
| | $\hat{\lambda} = 0.0135$ | | | | | |
| **BeNH** | $\hat{\theta} = 0.4935$ | **102.14** | **212.28** | **213.17** | **219.93** | **215.19** |
| | $\hat{\gamma} = 2.0363$ | | | | | |
| | $\hat{\alpha} = 4.3564$ | | | | | |
| | $\hat{\lambda} = 0.0150$ | | | | | |
| **KuNH** | $\hat{\theta} = 0.5456$ | **101.74** | **211.49** | **212.38** | **219.14** | **214.40** |
| | $\hat{\gamma} = 2.1430$ | | | | | |
| | $\hat{\alpha} = 4.4613$ | | | | | |
| | $\hat{\lambda} = 0.0129$ | | | | | |
| **EGNH** | $\hat{\theta} = 0.1165$ | **101.57** | **211.15** | **212.04** | **218.80** | **214.06** |
| | $\hat{\gamma} = 0.8936$ | | | | | |
| | $\hat{\alpha} = 0.6071$ | | | | | |
| | $\hat{\lambda} = 14.782$ | | | | | |
| **WeNH** | $\hat{\theta} = 0.5542$ | **101.30** | **210.60** | **211.49** | **218.25** | **213.51** |
| | $\hat{\gamma} = 0.4565$ | | | | | |
| | $\hat{\alpha} = 11.352$ | | | | | |
| | $\hat{\lambda} = 0.0114$ | | | | | |
| **LGamNH** | $\hat{\theta} = 0.4960$ | **102.14** | **212.29** | **213.17** | **219.93** | **215.20** |
| | $\hat{\gamma} = 1.8787$ | | | | | |
| | $\hat{\alpha} = 4.5703$ | | | | | |
| | $\hat{\lambda} = 0.0139$ | | | | | |
| **NH** | $\hat{\theta} = 0.3470$ | **103.12** | **210.24** | **210.49** | **214.06** | **211.69** |
| | $\hat{\gamma} = 3.2305$ | | | | | |

**Table 2.**

**Table2: Results of A and W for data 1**

| Distribution | W | A |
|---|---|---|
| **GoNH** | **0.1217** | **0.7687** |
| **[0,1]TEENH** | **0.1511** | **0.9610** |
| **BeNH** | **0.1413** | **0.9280** |
| **KuNH** | **0.1367** | **0.8876** |
| **EGNH** | **0.1450** | **0.9032** |

| | | |
|---|---|---|
| **WeNH** | **0.1285** | **0.8443** |
| **LGamNH** | **17.72** | **100.02** |
| **NH** | **0.1763** | **1.0948** |

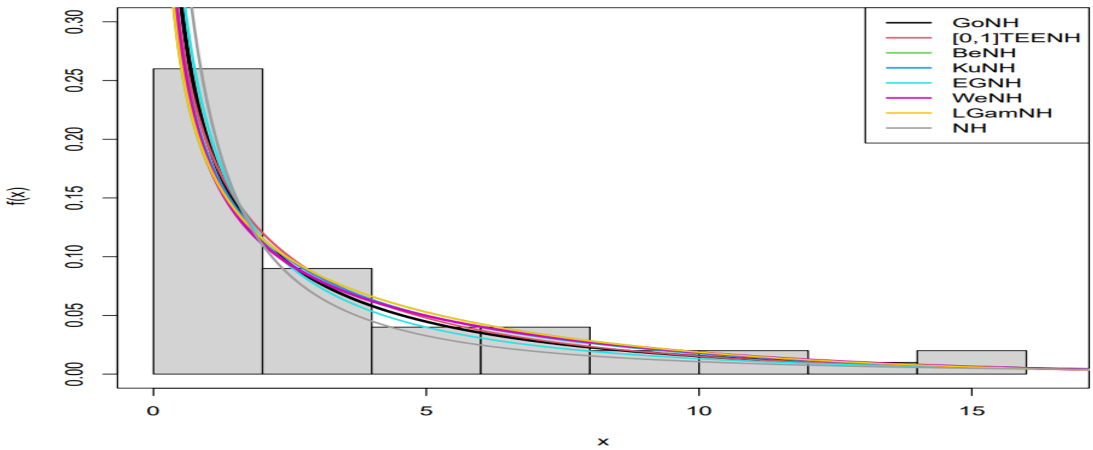


**FIGURE 5. Charts of the fitted distribution**

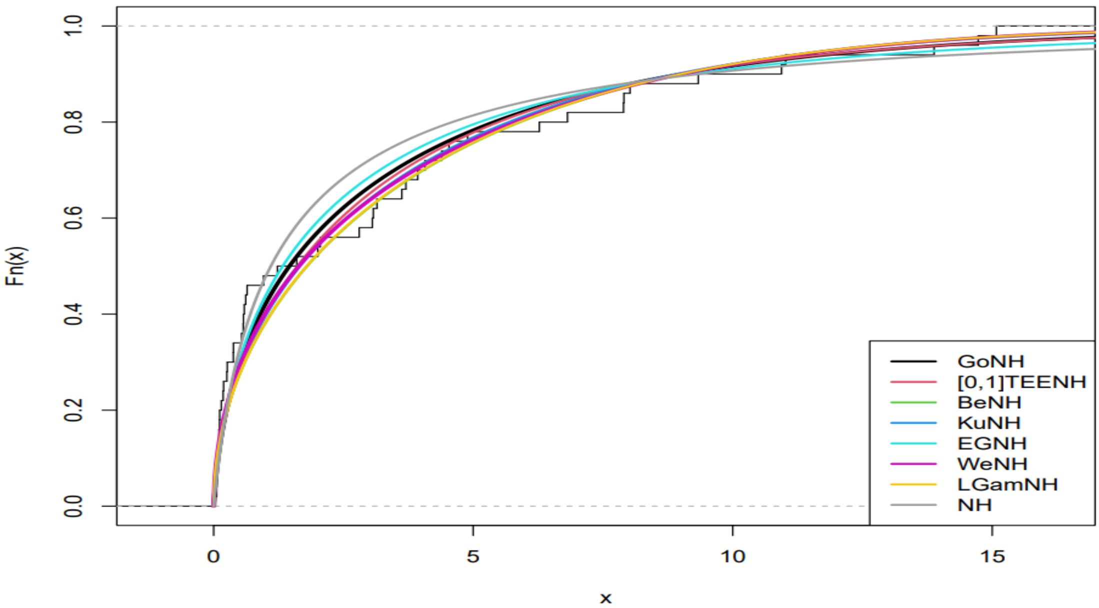


**FIGURE 6. Emperimental cdf of the fitted distribution**

Table 3. **Estimates of models for data 2**

| Distribution | Estimates | NLL | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|---|
| **GoNH** | $\hat{\theta}$ =**0.0303**<br>$\hat{\gamma}$ =**1.3380**<br>$\hat{\alpha}$ =**1.5179**<br>$\hat{\lambda}$ =**0.1991** | **79.70** | **167.40** | **168.54** | **174.15** | **169.84** |
| **[0,1]TEENH** | $\hat{\theta}$ =**18.4020**<br>$\hat{\gamma}$ =**9.6556**<br>$\hat{\alpha}$ =**2.1396**<br>$\hat{\lambda}$ =**0.0119** | **90.85** | **189.70** | **190.84** | **196.46** | **192.14** |

| | | | | | | |
|---|---|---|---|---|---|---|
| **BeNH** | $\hat{\theta}$ =4.5763<br>$\hat{\gamma}$ = 1.0291<br>$\hat{\alpha}$ =10.2139<br>$\hat{\lambda}$ =0.0182 | **84.90** | **177.80** | **178.94** | **184.55** | **180.24** |
| **KuNH** | $\hat{\theta}$ =4.0797<br>$\hat{\gamma}$ =19.7925<br>$\hat{\alpha}$ =7.4624<br>$\hat{\lambda}$ =0.0100 | **83.22** | **174.44** | **175.58** | **181.20** | **176.88** |
| **EGNH** | $\hat{\theta}$ =0.0924<br>$\hat{\gamma}$ =2.5328<br>$\hat{\alpha}$ =3.7278<br>$\hat{\lambda}$ =0.1823 | **83.22** | **174.44** | **175.59** | **181.20** | **176.89** |
| **WeNH** | $\hat{\theta}$ =1.7953<br>$\hat{\gamma}$ =9.4046<br>$\hat{\alpha}$ =16.5076<br>$\hat{\lambda}$ =0.0215 | **80.33** | **168.66** | **169.80** | **175.42** | **171.10** |
| **LGamNH** | $\hat{\theta}$ =8.8116<br>$\hat{\gamma}$ =0.3700<br>$\hat{\alpha}$ =0.9184<br>$\hat{\lambda}$ =5.1587 | **87.67** | **183.34** | **184.48** | **190.09** | **185.78** |
| **NH** | $\hat{\theta}$ =26.5983<br>$\hat{\gamma}$ =0.0045 | **102.15** | **208.30** | **208.62** | **211.68** | **209.52** |

Table 4. **Table of Results W and A for data 1**

| Distribution | W | A |
|---|---|---|
| **GoNH** | **0.1217** | **0.7687** |
| **[0,1]TEENH** | **0.1511** | **0.9610** |
| **BeNH** | **0.1413** | **0.9280** |
| **KuNH** | **0.1367** | **0.8876** |
| **EGNH** | **0.1450** | **0.9032** |
| **WeNH** | **0.1285** | **0.8443** |
| **LGamNH** | **17.722** | **100.02** |
| **NH** | **0.1763** | **1.0948** |

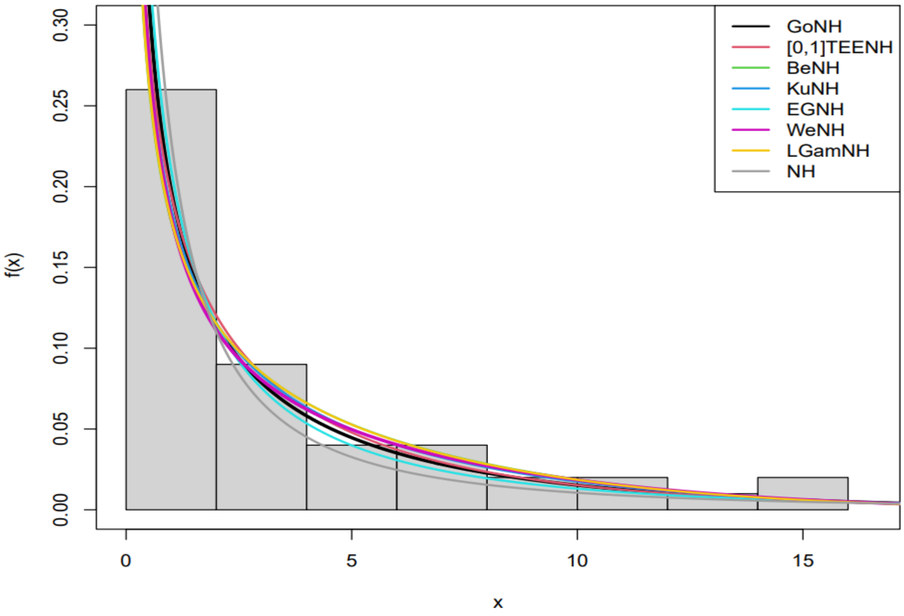


**FIGURE 7. Charts of the fitted distribution**

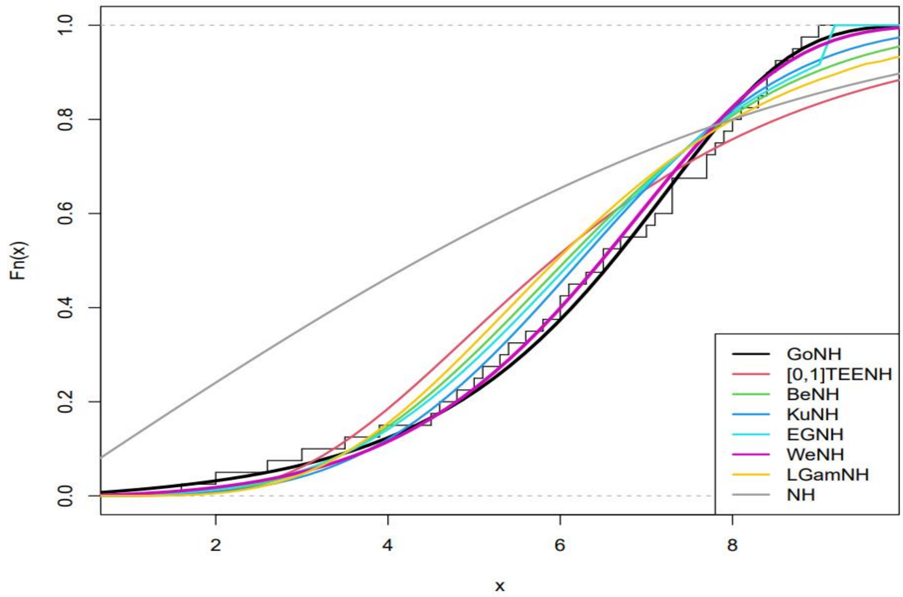


**FIGURE 8. Emperimental cdf of the fitted distribution**

The GoNH distribution has the lowest values for A, W, HQIC, BIC, CAIC and AIC (for the two real data sets), according to the values in Tables 1, 2, 3 and 4. Thus, the GoNH model can be selected as the ideal model. Figures 5, 6, 7 and 8 show the estimated pdfs and cdfs plots. Figures 5, 6, 7 and 8 make it evident that the revised GoNH distribution best matches the two datasets.

## 6. Conclusions

This study proposed a new continuous statistical distribution class, namely the GoNH distribution. Explicit formulas for moments, moment-generating functions, survival functions, risk functions, characteristic functions, quantity functions, pdf expansions and ordered statistics are just a few of the mathematical features. And the many statistics of the recently proposed model. The maximum likelihood technique was used to estimate the parameters of the model. To determine the extent of flexibility in the new distribution, two groups were used. The first set consists of the failure frequency of 50 components, while the second set comprises 40 observations. It has been studied in many research studies using real engineering data. The GoNH distribution was compared with different distributions ([0,1]TEENH, BeNH, KuNH, EGNH, WeNH, LGamNH and NH) using certain criteria. Based on the statistical results, it was found that the new distribution is more suitable for the data compared to other distributions. The closest distribution for the first group was NH, while for the second group, it was WeNH. The figures presented in the study showed that the GoNH distribution provided a superior fit, supporting the numerical results from the data.

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## CONFLICTS OF INTEREST

None

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