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Approximation of an Inertial Iteration Method for a Set-Valued Quasi Variational Inequality

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ABSTRACT: In this article, a projection type two step inertial iterative scheme for investigating set-valued quasi variational inequality in real Banach spaces is designed. We manifest the existence result and verified by an illustrative example. Also, we estimate the approximate solution of a set-valued quasi variational inequality by analyzing the convergence of the proposed inertial iterative algorithm. Further, an extended Weiner-Hopf equation is considered and substantiated that it is analogous to the extended set-valued quasi variational inequality. Finally, we investigate the extended Weiner-Hopf equation by analyzing the convergence of the composed iterative scheme.

Keywords: Set-valued quasi variational inequality; inertial iterative algorithm; strong convergence; Weiner-Hopf equation

1. INTRODUCTION

Assume that $\Omega \neq \phi$ is a closed convex set in a Banach space $\mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{B}$ be a nonlinear mapping. The variational inequality problem (VIP) is to observe a component $\hat{a} \in \Omega$ so that

$$\langle \varphi(\hat{a}), \hat{b} - \hat{a} \rangle \geq 0, \forall \hat{b} \in \Omega. \tag{1}$$

It is commonly known that VIP introduced by Stampacchia [50] is a highly effective and useful tool for analyzing issues that arise in all diverse areas of natural sciences. Because of applicability and fruitful outcomes, VIPs have been broadened and diversified in a number of ways, see, [5, 6, 14–16, 18, 29, 39]. Among these generalizations, quasi variational inequality $QVI(\Omega(\hat{a}), \varphi)$ is a prominent generalization which is to observe a component $\hat{a} \in \Omega(\hat{a})$ so that

$$\langle \varphi(\hat{a}), \hat{b} - \hat{a} \rangle \geq 0, \forall \hat{b} \in \Omega(\hat{a}), \tag{2}$$

where $\Omega : \mathcal{B} \to 2^{\mathcal{B}}$ be a set-valued mapping with $\Omega(\hat{a}) \subset \mathcal{B}, \forall \hat{a} \in \mathcal{B}$. If $\Omega(\hat{a}) = \Omega, \forall \hat{a} \in \mathcal{B}$, then $QVI(\Omega(\hat{a}), \varphi)$ reduces to VIP (1). The study of QVIs has been recognized an extremely practical and applicable field. A model of quasi variational inequalities has been used to frame a number of problems with practical applications, including free boundary problems, mechanics, economy, and stochastic impulsive control modeling, see, [8, 9, 12, 25, 26, 31]. It was demonstrated by Noor et al. [38, 41] that the obstacle boundary value problem ($OBVP$) of pointing out w so that

$$\begin{cases} -w'''(x) \geq \varphi(x), \text{ on } \mathcal{D} = [a_1, a_2] \\ w(x) \geq C(w), \text{ on } \mathcal{D} = [a_1, a_2] \\ [-w'''(x) - \varphi(x)][w - C(w)] = 0, \text{ on } \mathcal{D} = [a_1, a_2] \\ w(a_1) = 0, w'(a_1) = 0, w'(a_2) = 0, \end{cases} \tag{3}$$

where φ is continuous and $C(w)$ represents the obstacle (cost) function can be formulated as following generalized quasi variational inequality

$$\langle \theta(w), \phi(q) - \phi(w) \rangle \geq \langle \varphi, \phi(q) - \phi(w) \rangle, \forall q \in \Omega(w). \tag{4}$$

Following important lemma assures the existence of solution of $QVI(\Omega(\hat{a}), \varphi)$.

Lemma 1. *[37] Assume that the subsequent premises hold true.*

(i) Suppose $\varphi : \mathcal{H} \to \mathcal{H}$ *is* δ_φ*-Lipschitz continuous and* ς*-strongly monotone.*

(ii) The projection mapping $P_{\Omega(\hat{a})} : \mathcal{H} \to \Omega(\hat{a})$ *satisfies*

$$P_{\Omega(\hat{a})}(\xi) - P_{\Omega(\hat{b})}(\xi) \leq \nu \|\hat{a} - \hat{b}\|, \forall \hat{a}, \hat{b}, \xi \in \mathcal{H},$$

(iii) $\exists \hat{b} \geq 0$ *comply with* $\hat{b} + \sqrt{1 - (\varsigma/\delta_\varphi)^2} < 1$.

Then $QVI(\Omega(\hat{a}), \varphi)$ *has a unique solution.*

In case $\Omega(\hat{a}) = \Omega$, assumption (ii) holds for $\hat{b} = 0$ and the assumption (iii) becomes trivial. Therefore, assuming assumption (i) of Lemma 1 holds, then VIP (1.1) has a unique solution, see [37]. The $QVI(\Omega(\hat{a}), \varphi)$ was initially developed by Bensoussan and Lions [9] for addressing problems related to impulse control. QVIs bring forth an integrated framework for variation inequalities and integrated modeling of significant physical problems. Several applications and significance of QVIs include superconductivity, thermoplasticity or electrostatics [22, 23, 28], continuum and solid mechanics [10, 27, 42], transportation [11, 47], game theory [21], etc.

However, the fixed point theory has become fastest expanding research field. Numerous problems appearing in science, and engineering, particularly in ODEs, PDEs, VIs, and zeros of monotone operators have been investigated by transforming as a model of fixed point problem. The fixed point of a nonlinear mapping $\varphi : \mathcal{B} \to \mathcal{B}$ is described as $Fix(\varphi) = \{\hat{a} \in \mathcal{B} : \varphi(\hat{a}) = \hat{a}\}$. Owing to the significance of fixed points numerous new iterative schemes have been designed and tackled over the last few years. The extensively researched and widely used method for determining fixed points is due to Mann iteration method [35], which is given as:

$$\mu_{n+1} = (1 - \alpha_n)\mu_n + \alpha_n \varphi(\mu_n), n \in \mathbb{N}, \tag{5}$$

where $\alpha_n \in [0, 1]$.. A few common and widely used iteration techniques include Ishikawa iteration [24], Halpern iteration [20], S-iteration [2].

Researchers are fascinated by the possibility of achieving enhanced convergence rates by developing iterative methods to obtain the solutions of nonlinear problems. In an attempt to obtain an increasing rate of convergence, several iterative schemes have been investigated and tested. A such attempt was made to enhance the convergence rate by adding inertial term, see, [29, 34, 45, 46, 49]. It was Polyak [43] who originated the inertial term to investigate an optimization problem. In order to acquire the inertial term the heavy ball method was employed by discretizing the following second order system:

$$\mu''(t) + \psi \mu'(t) + \nabla \omega(\mu(t)) = 0, \tag{6}$$

where $\omega : \mathcal{H} \to \mathbb{R}$ is differentiable, $\mu(t)$ represents time continuous trajectory, $\psi(\mu(t))$ external gravitational field and $\psi > 0$ the friction. Richardson [44] employed a relaxation method to solve a linear systems with augmented rate of convergence. In this sequel, Eckstein and Bertsekas [17] accelerated a proximal point algorithm by adding an relaxation parameter. Further by adding relaxation techniques with inertial term, Alvarez [7] proposed an iteration process to deal with convex optimization and monotone inclusion problems. In this development, Maigne [32] devised the inertial Mann iterative scheme to reckon fixed points of a nonexpansive mapping as:

$$\begin{cases} \nu_n = \mu_n + \tau_n(\mu_n + \mu_{n-1}), \\ \mu_{n+1} = (1 - a_n)\nu_n + a_n \varphi \nu_n, \forall n \in \mathbb{N}, \end{cases} \tag{7}$$

where a_n is a relaxation factor and τ_n is a damping term. Owing to the attraction of researchers and importance of augmented rate of convergence, so far numerous iterative methods involving single or double inertial terms have been designed and analyzed, see [1, 3, 4, 19, 48, 51**?**]. Recently, Çopur et al. [13] investigated $QVI(\Omega(\hat{a}), \varphi)$ by analyzing the following inertial iteration process involving two inertial terms as follows:

$$\begin{cases} \mu_n = w_n + \gamma_n(\zeta_n - \zeta_{n-1}), \\ \nu_n = w_n + \tau_n(\zeta_n - \zeta_{n-1}), \\ \zeta_{n+1} = (1 - a_n - b_n)\nu_n + a_n \varphi(\nu_n) + b_n \mu_n, \end{cases} \tag{8}$$

where $\{\gamma_n\},\{\tau_n\}$ are sequences in $(0,1)$. Motivated and convinced by the acknowledged information cited in the sources mentioned above, our motive is to examine the $EQVI(\mathcal{B},\psi,\varphi,\Psi)$. We manifest the existence result which is validated by an illustrative example. Also, we estimate the approximate solution of $EQVI(\mathcal{B},\psi,\varphi,\Psi)$ by analyzing the convergence of two steps inertial iterative algorithm based on (8). Further, an extended Weiner-Hopf equation is considered and substantiated that it is analogous to $EQVI(\mathcal{B},\psi,\varphi,\Psi)$. By implementing this equivalence, we derive the solution of $EQVI(\mathcal{B},\psi,\varphi,\Psi)$. Lastly, we investigate the extended Weiner-Hopf equation by analyzing the convergence of the composed scheme.

2. PRELUDES AND EXISTENCE RESULTS

Let $\mathcal{B}$ be a real Banach space equipped with norm $\|\cdot\|$; $\langle\cdot,\cdot\rangle$ is the duality pairing between $\mathcal{B}$ and its dual space $\mathcal{B}^*$; the family of closed and bounded subsets of $\mathcal{B}$ is represented by $CB(\mathcal{B})$ and $2^{\mathcal{B}}$ is the power set of $\mathcal{B}$. The normalized duality mapping $J:\mathcal{B}\to 2^{\mathcal{B}^*}$ is expressed as

$$J(\hat{a}) = \{\varphi\in\mathcal{B}^* : \langle\hat{a},\varphi\rangle = \|\hat{a}\|^2 = \|\varphi\|^2\}, \forall \hat{a}\in\mathcal{B}.$$

Lemma 2. *For all $e_1,e_2\in\mathcal{B}$, $J:\mathcal{B}\to\mathcal{B}^*$ is characterized by the following inequalities:*

(i) $\|e_1+e_2\|^2 \le \|e_1\|^2 + 2\langle e_2, J(e_1+e_2)\rangle$;

(ii) $\langle e_1-e_2, Je_1-Je_2\rangle \le 2d^2\rho_{\mathcal{B}}(4\|e_1-e_2\|/d)$;

where, $d=\sqrt{(\|e_1\|^2+\|e_2\|^2)/2}$ and the modulus of smoothness of $\mathcal{B}$ is expressed as

$$\rho_{\mathcal{B}}(t) = \sup\Big\{\frac{\|e_1+e_2\|+\|e_1-e_2\|}{2} - 1 : \|e_1\|\le 1, \|e_2\|\le t\Big\}.$$

The Hausdorff metric $\mathcal{D}(\cdot,\cdot)$ on $CB(\mathcal{B})$ is expressed as

$$\mathcal{D}(\varphi,\psi) = \max\Big\{\sup_{e_1\in\varphi}\inf_{e_2\in\psi} d(e_1,e_2), \sup_{e_2\in\psi}\inf_{e_1\in\varphi} d(e_1,e_2)\Big\}.$$

Now onward, the Banach space $\mathcal{B}$ is taken as a uniformly smooth. Let $\Omega\neq\phi$ be a closed convex subset of $\mathcal{B}$. A mapping $R_\Omega:\mathcal{B}\to\Omega$ is retraction if $R_\Omega^2 = R_\Omega$, nonexpansive retraction if R_Ω is retraction and $\|R_\Omega(e_1)-R_\Omega(e_2)\|\le\|e_1-e_2\|, \forall e_{12}, e_2\in\mathcal{B}$; and sunny retraction if $R_\Omega(R_\Omega e_1 - t(e_1-R_\Omega e_1) = R_\Omega e_1), \forall e_1\in\mathcal{B}, t\in\mathbb{R}$. The following lemma is essential and necessary to accomplishing the goal.

Lemma 3. *A mapping $R_\Omega:\mathcal{B}\to\Omega$ is sunny nonexpansive retraction if and only if*

$$\langle e_1 - R_\Omega(e_1), J(R_\Omega(e_1)-e_2)\rangle \ge 0, \forall e_1\in\mathcal{B}, e_2\in\Omega(e_1).$$

Assumption C [40] For given $e_1,e_2,\varsigma^*\in\mathcal{B}$ and constant $\kappa>0$, the mapping P_Ω fulfills the condition

$$\|P_{\Omega(e_1)}(\varsigma^*) - P_{\Omega(e_2)}(\varsigma^*)\| \le \kappa\|e_1-e_2\|.$$

Definition 1. *A mapping $\varphi:\mathcal{B}\to\mathcal{B}$ is referred as*

(i) η-strongly accretive, if $\exists\eta\ge 0$,

$$\langle\varphi(e_1)-\varphi(e_2), J(e_1-e_2)\rangle \ge \eta\|e_1-e_2\|^2, \forall e_1,e_2\in\mathcal{B};$$

(ii) δ-Lipschitz continuous, if $\exists\delta>0$,

$$\|\varphi(e_1)-\varphi(e_2)\| \le \delta\|e_1-e_2\|, \forall e_1,e_2\in\mathcal{B},$$

(iii) κ-expanding if, $\exists\kappa>0$,

$$\|\varphi(e_1)-\varphi(e_2)\| \ge \kappa\|e_1-e_2\|, \forall e_1,e_2\in\mathcal{B}.$$

Note that η-strongly accretive mapping φ is η-expanding.

Definition 2. *Let $\varphi,S,T:\mathcal{B}\to\mathcal{B}$; $\psi:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ be the single-valued mappings. Then $\psi(S,T)$ is referred as* *(i) τ-strongly accretive regarding φ, if $\exists\tau \geq 0$,*

$$\langle \psi(S(e_1), T(e_1)) - \psi(S(e_2), T(e_2)), J(\varphi(e_1) - \varphi(e_2)\rangle \geq \tau\|e_1 - e_2\|^2, \forall e_1, e_2 \in \mathcal{B};$$

(ii) ς-Lipschitz continuous, if $\exists\varsigma > 0$,

$$\|\psi(S(e_1), T(e_1)) - \psi(S(e_2), T(e_2))\| \leq \varsigma\|e_1 - e_2\|, \forall e_1, e_2 \in \mathcal{B}.$$

Definition 3. *A mapping $\psi : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ is called (σ, υ)-mixed Lipschitz continuous, if for some $\sigma, \upsilon > 0$,*

$$\|\psi(e_1, t_1) - \psi(e_2, t_2)\| \leq \sigma\|e_1 - e_2\| + \upsilon\|t_1 - t_2\|, \forall e_1, e_2, t_1, t_2 \in \mathcal{B}.$$

Remark 1. *Let $A : \mathcal{B} \to CB(\mathcal{B})$ be a set-valued mapping, then for every $\hat{a}, \hat{b} \in \mathcal{B}$ and $\mu \in A(e_1), \upsilon \in B(e_2)$ there exist $\epsilon, \delta > 0$ such that*

(i) $\|\mu - \upsilon\| \leq \mathcal{D}(A(e_1), A(e_2)) + \epsilon\|e_1 - e_2\|$,

(ii) $\|\mu - \upsilon\| \leq \delta\mathcal{D}(A(e_1), A(e_2))$.

Lemma 4. *[30] Suppose that the nonnegative real sequences $\{\hat{a}_n\}$ and $\{\hat{b}_n\}$ satisfy*

$$\hat{a}_{n+1} \leq \tau\hat{a}_n + \hat{b}_n, \forall n \in \mathbb{N},$$

for some $0 \leq \tau < 1$. If $\lim_{n\to\infty} \hat{b}_n = 0$, then $\lim_{n\to\infty} \hat{a}_n = 0$,

Next, we shall define $EQVI(\mathcal{B}, \psi, \varphi, \Psi)$. Let $A, B : \mathcal{B} \to CB(\mathcal{B})$ be set-valued mappings, $\Psi, \psi : \mathcal{B}\times\mathcal{B} \to \mathcal{B}, \varphi, S, T : \mathcal{B} \to \mathcal{B}$ be single-valued mappings, and for any $\hat{a} \in \mathcal{B}$, $\Omega : \mathcal{B} \to 2^{\mathcal{B}}$ assigns a closed convex-valued set $\Omega(\hat{a})$ in $\mathcal{B}$. We are intended to examine the extended quasi variational inequality $EQVI(\mathcal{B}, \psi, \varphi, \Psi)$ which is to find $\{(\hat{a}, \mu, \upsilon) : \hat{a} \in \mathcal{B}, \varphi(\hat{a}) \in \Omega(\hat{a}), \mu \in A(\hat{a}), \upsilon \in B(\hat{a})\}$ such that

$$\langle \rho(\psi(S(\hat{a}), T(\hat{a})) + \varphi(\hat{a})) - \Psi(\mu, \nu), J(\varphi(\hat{b}) - \varphi(\hat{a}))\rangle \geq 0, \forall \hat{b} \in \mathcal{B}, \varphi(\hat{b}) \in \Omega(\hat{a}). \tag{9}$$

$EQVI(\mathcal{B}, \psi, \varphi, \Psi)$ is a broader and unified class of quasi variational inequality. Several VIs and QVIs can be achieved by $EQVI(\mathcal{B}, \psi, \varphi, \Psi)$ for different selection of involved mappings. In the next result, we obtain an equivalent fixed point problem by transforming ($EQVI(\mathcal{B}, \psi, \varphi, \Psi)$).

Lemma 5. *Let $P_{\Omega(\hat{a})} : \mathcal{B} \to \Omega(\hat{a})$ be a sunny nonexpansive retraction. An element $(\hat{a}, \mu, \nu)$ with $\hat{a} \in \mathcal{B}, \varphi(\hat{a}) \in \Omega, \mu \in A(\hat{a}), \nu \in B(\hat{a})$ solves $EQVI(\mathcal{B}, \psi, \varphi, \Psi)$ if and only if $\hat{a} \in Fix(Q)$, where*

$$Q(\hat{a}) = \hat{a} - \varphi(\hat{a}) + P_{\Omega(\hat{a})}[(1 - \rho)\varphi(\hat{a}) - \rho\psi(S(\hat{a}), T(\hat{a})) + \Psi(\mu, \nu)], \tag{10}$$

and $\rho > 0$ is a constant.

Theorem 1. *Suppose that $\varphi : \mathcal{B} \to \mathcal{B}$ be κ-strongly accretive, δ-Lipschitz continuous mapping; $\Psi, \psi : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ and $S, T : \mathcal{B} \to \mathcal{B}$ be the single-valued mappings so that $\psi(S, T)$ is τ-strongly accretive regarding φ and ς-Lipschitz continuous, Ψ is (σ, υ)-mixed Lipschitz continuous and $A, B : \mathcal{B} \to CB(\mathcal{B})$ be θ-$\mathcal{D}$-Lipschitz continuous, ϑ-$\mathcal{D}$-Lipschitz continuous, respectively. If the retraction $P_{\Omega(\hat{a})} : \mathcal{B} \to \Omega(\hat{a})$ comply with the Assumption C and constant $\rho > 0$ satisfies:*

$$\Theta + \rho\delta < 1 - \Phi, \kappa < \frac{1 + 64\varepsilon\delta^2}{2}, \tau < \frac{\delta^2 + 64\varepsilon\varsigma^2}{2\rho}. \tag{11}$$

Then $(\hat{a}, \mu, \nu)$ such that $\hat{a} \in \mathcal{B}, \varphi(\hat{a}) \in \Omega(\hat{a}), \mu \in A(\hat{a}), \nu \in B(\hat{a})$ solves $EQVI(\mathcal{B}, \psi, \varphi, \Psi)$.

Proof. Assume that for given $\hat{a}, \hat{b} \in \mathcal{B}, \mu \in A(\hat{a}), \nu \in B(\hat{a}), \bar{\mu} \in A(\hat{b}), \bar{\nu} \in B(\hat{b})$, there exist $a^* \in Q(\hat{a})$ and $b^* \in Q(\hat{b})$ so that

$$a^* = \hat{a} - \varphi(\hat{a}) + P_{\Omega(\hat{a})}[(1 - \rho)\varphi(\hat{a}) - \rho\psi(S(\hat{a}), T(\hat{a})) + \Psi(\mu, \nu)], \tag{12}$$

and

$$b^* = \hat{b} - \varphi(\hat{b}) + P_{\Omega(\hat{b})}[(1 - \rho)\varphi(\hat{b}) - \rho\psi(S(\hat{b}), T(\hat{b})) + \Psi(\bar{\mu}, \bar{\nu})]. \tag{13}$$

By (12), (13) and considering the Assumption C, we obtain

$$\begin{aligned}
\|a^* - b^*\| &\leq \|\hat{a} - \hat{b} - (\varphi(\hat{a}) - \varphi(\hat{b}))\| \\
&\quad + \|P_{\Omega(\hat{a})}[(1-\rho)\varphi(\hat{a}) - \rho\psi(S(\hat{a}), T(\hat{a})) + \Psi(\mu, \nu)] \\
&\quad - P_{\Omega(\hat{b})}[(1-\rho)\varphi(\hat{b}) - \rho\psi(S(\hat{b}), T(\hat{b})) + \Psi(\bar{\mu}, \bar{\nu})]\| \\
&\leq \|\hat{a} - \hat{b} - (\varphi(\hat{a}) - \varphi(\hat{b}))\| + \xi\|\hat{a} - \hat{b}\| \\
&\quad + \|(1-\rho)\varphi(\hat{a}) - \rho\psi(S(\hat{a}), T(\hat{a})) + \Psi(\mu, \nu) \\
&\quad - [(1-\rho)\varphi(\hat{b}) - \rho\psi(S(\hat{b}), T(\hat{b})) + \Psi(\bar{\mu}, \bar{\nu})]\| \\
&\leq \|\hat{a} - \hat{b} - (\varphi(\hat{a}) - \varphi(\hat{b}))\| + \xi\|\hat{a} - \hat{b}\| + \rho\|\varphi(\hat{a}) - \varphi(\hat{b})\| \\
&\quad + \|\varphi(\hat{a}) - \varphi(\hat{b}) - \rho(\psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b})))\| \\
&\quad + \|\Psi(\mu, \nu) - \Psi(\bar{\mu}, \bar{\nu})\|.
\end{aligned} \tag{14}$$

Employing κ-strongly accretive property, δ-Lipschitz continuity of φ and taking advantage of Lemma 2, we achieve

$$\begin{aligned}
&\|\hat{a} - \hat{b} - (\varphi(\hat{a}) - \varphi(\hat{b}))\|^2 \\
&\leq \|\hat{a} - \hat{b}\|^2 - 2\langle \varphi(\hat{a}) - \varphi(\hat{b}), J(\hat{a} - \hat{b} - (\varphi(\hat{a}) - \varphi(\hat{b})))\rangle \\
&\leq \|\hat{a} - \hat{b}\|^2 - 2\langle \varphi(\hat{a}) - \varphi(\hat{b}), J(\hat{a} - \hat{b})\rangle \\
&\quad + 2\langle \varphi(\hat{a}) - \varphi(\hat{b}), J(\hat{a} - \hat{b} - (\varphi(\hat{a}) - \varphi(\hat{b}))) - J(\hat{a} - \hat{b})\rangle \\
&\leq (1 - 2\kappa + 64\varepsilon\delta^2)\|\hat{a} - \hat{b}\|^2.
\end{aligned} \tag{15}$$

Recalling the Lemma 2 and assumptions that φ is δ-Lipschitz continuous, $\psi(S, T)$ is τ-strongly accretive regarding φ and ς-Lipschitz continuous, we achieve

$$\begin{aligned}
&\|\varphi(\hat{a}) - \varphi(\hat{b}) - \rho(\psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b})))\|^2 \\
&\leq \|\varphi(\hat{a}) - \varphi(\hat{b})\|^2 - 2\rho\langle \psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b})), \\
&\quad J(\varphi(\hat{a}) - \varphi(\hat{b}) - \rho(\psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b}))))\rangle \\
&\leq \|\varphi(\hat{a}) - \varphi(\hat{b})\|^2 - 2\rho\langle \psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b}))), \\
&\quad J(\varphi(\hat{a}) - \varphi(\hat{b}))\rangle + 2\rho\langle \psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b}))), \\
&\quad J(\varphi(\hat{a}) - \varphi(\hat{b}) - \rho(\psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b}))) \\
&\quad - J(\varphi(\hat{a}) - \varphi(\hat{b}))\rangle \\
&\leq \delta^2\|\hat{a} - \hat{b}\|^2 - 2\rho\tau\|\hat{a} - \hat{b}\|^2 + 64\varepsilon\varsigma^2\|\hat{a} - \hat{b}\|^2 \\
&= (\delta^2 - 2\rho\tau + 64\varepsilon\varsigma^2)\|\hat{a} - \hat{b}\|^2.
\end{aligned} \tag{16}$$

Employing (σ, υ)-mixed Lipschitz continuity of Ψ, Lipschitz continuities of A and B yields

$$\begin{aligned}
\|\Psi(\mu, \nu) - \Psi(\bar{\mu}, \bar{\nu})\| &\leq \sigma\|\mu - \bar{\mu}\| + \upsilon\|\nu - \bar{\nu}\| \\
&\leq \sigma\mathcal{D}(A(\hat{a}), A(\hat{b})) + \upsilon\mathcal{D}(B(\hat{a}), B(\hat{b})) \\
&\leq \sigma\theta\|\hat{a} - \hat{b}\| + \upsilon\vartheta\|\hat{a} - \hat{b}\| \\
&= (\sigma\theta + \upsilon\vartheta)\|\hat{a} - \hat{b}\|.
\end{aligned} \tag{17}$$

Combining (15), (16) and (17), (14) turns into

$$\begin{aligned}
\|a^* - b^*\| &\leq \{\sqrt{1 - 2\kappa + 64\varepsilon\delta^2} + (\xi + \rho\delta) \\
&\quad + \sqrt{\delta^2 - 2\rho\tau + 64\varepsilon\varsigma^2} + (\sigma\theta + \upsilon\vartheta)\}\|\hat{a} - \hat{b}\| \\
&= (\Theta + \rho\delta + \Phi)\|\hat{a} - \hat{b}\| = L\|\hat{a} - \hat{b}\|,
\end{aligned} \tag{18}$$

where,

$$L = \Theta + \rho\delta + \Phi, \Theta = \sqrt{1 - 2\kappa + 64\varepsilon\delta^2} + \sigma\theta + \upsilon\vartheta, \Phi = \xi + \sqrt{\delta^2 - 2\rho\tau + 64\varepsilon\varsigma^2}. \tag{19}$$

Clearly, from (18) and condition (11), we conclude that the set-valued mapping Q is a contraction and hence $\exists\hat{a} \in \mathcal{B}$ so that $\hat{a} \in Fix(Q)$. Thus, Lemma 5 guarantees that $(\hat{a}, \mu, \upsilon)$ so that $\hat{a} \in \mathcal{B}, \mu \in A(\hat{a}), \upsilon \in B(\hat{a})$ solves $EQVI(\mathcal{B}, \psi, \varphi, \Psi)$.

Example 1. *Consider a real Banach space* $l_2 = \{\hat{a} = (\hat{a}_0, \hat{a}_1, \hat{a}_2, \cdots) : \sum_{n=0}^{\infty} |\hat{a}_n|^2 < \infty, \hat{a}_n \in \mathbb{R}, \forall n = 0, 1, 2, \cdots\}$ *and* $\|\hat{a}\|_2 = \left(\sum_{n=0}^{\infty} |\hat{a}_n|^2\right)^{1/2}$. *Define* $\varphi, S, T : \mathcal{B} \to \mathcal{B}; \psi, \Psi : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ *and* $A, B : \mathcal{B} \to CB(\mathcal{B})$ *by*

$$\varphi(\hat{a}) = \frac{\hat{a}}{2}, S(\hat{a}) = \frac{\hat{a}}{4}, T(\hat{a}) = \frac{\hat{a}}{8}, \forall \hat{a} \in \mathcal{B},$$

$$\psi(S(\hat{a}), T(\hat{a})) = \frac{(S(\hat{a}) + T(\hat{a}))}{4}, \Psi(\mu, \nu) = \frac{\mu + \nu}{8}$$

$$\text{and } A(\hat{a}) = \left\{\frac{\hat{a}}{4}\right\}, \quad B(\hat{a}) = \left\{\frac{2\hat{a}}{3}\right\}.$$

Then, one can discern that for all $\hat{a}, \hat{b} \in \mathcal{B}$,

$$\langle \hat{a} - \hat{b}, \varphi(\hat{a}) - \varphi(\hat{b}) \rangle = \langle \hat{a} - \hat{b}, \frac{\hat{a}}{2} - \frac{\hat{b}}{2} \rangle = \frac{1}{2}\|\hat{a} - \hat{b}\|_2^2,$$

$$\|\varphi(\hat{a}) - \varphi(\hat{b})\|_2 = \|\frac{1}{2}\hat{a} - \frac{1}{2}\hat{b}\|_2 = \frac{1}{2}\|\hat{a} - \hat{b}\|_2.$$

i.e., φ *is* $\frac{1}{2}$*-strongly accretive and* $\frac{1}{2}$*-Lipschitz continuous. Also,*

$$\langle \psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b})), \varphi(\hat{a}) - \varphi(\hat{b}) \rangle = \langle \frac{3\hat{a}}{32} - \frac{3\hat{b}}{32}, \frac{\hat{a}}{2} - \frac{\hat{b}}{2} \rangle = \frac{3}{64}\|\hat{a} - \hat{b}\|_2^2,$$

$$\|\psi(S(\hat{a}), T(\hat{a})) - \psi(S(\hat{b}), T(\hat{b}))\|_2 = \left\|\frac{3\hat{a}}{32} - \frac{3\hat{b}}{32}\right\|_2 = \frac{3}{32}\|\hat{a} - \hat{b}\|_2.$$

i.e., $\psi(S, T)$ *is* $\frac{3}{64}$*-strongly accretive and* $\frac{3}{32}$*-Lipschitz continuous. Now, for all* $\mu \in A(\hat{a}), \bar{\mu} \in A(\hat{b}), \nu \in B(\hat{a}), \bar{\nu} \in B(\hat{b})$, *one can acquire*

$$\|\Psi(\mu, \nu) - \Psi(\bar{\mu}, \bar{\nu})\|_2 = \left\|\frac{\mu + \nu}{4} - \frac{\bar{\mu} + \bar{\nu}}{4}\right\|_2 \le \frac{1}{4}\|\mu - \bar{\mu}\|_2 + \frac{1}{4}\|\nu - \bar{\nu}\|_2,$$

$$\|\mu - \bar{\mu}\|_2 = \|\frac{\hat{a}}{4} - \frac{\hat{b}}{4}\|_2 = \frac{1}{4}\|\hat{a} - \hat{b}\|_2 \text{ and } \|\nu - \bar{\nu}\|_2 = \|\frac{2\hat{a}}{3} - \frac{2\hat{b}}{3}\|_2 = \frac{2}{3}\|\hat{a} - \hat{b}\|_2.$$

Thus, Ψ *is* $\left(\frac{1}{4}, \frac{1}{4}\right)$*-mixed Lipschitz continuous,* A *and* B *are* $\frac{1}{4}$ *and* $\frac{2}{3}$*-Lipschitz continuous, respectively. Define* $\Omega : \mathcal{B} \to \mathcal{B}$ *by*

$$\Omega(\hat{a}) = \Omega(\{\hat{a}_n\}) = \left\{t = \{t_n\} : t_0 \ge \frac{\hat{a}_0}{15}, t_n = 0, \forall n \in \mathbb{N}\right\}.$$

We assert that $\Omega(\hat{a})$ *is a closed and convex. For arbitrary* $\alpha \in [0, 1]$ *and* $t_0, m_0 \in \Omega(\hat{a})$, *we have* $\alpha t_0 + (1 - \alpha)m_0 \ge \frac{\hat{a}_0}{15}$ *and hence* $\Omega(\hat{a})$ *is a convex set. Define* $f : [\frac{\hat{a}_0}{15}, \infty) \to \Omega(\hat{a})$ *by* $f(r) = (r, 0, 0, \cdots)$. *Evidently,* f *is well defined. For* $t \ne m \in [\frac{\hat{a}_0}{15}, \infty)$, *we achieve* $(t, 0, 0, \cdots) \ne (m, 0, 0, \cdots)$, *i.e.,* f *is one-to-one. Simple observation reveals that there is an* $t_0 \in [\frac{\hat{a}_0}{15}, \infty)$ *so that* $f(t_0) = (t_0, 0, 0, \cdots)$ *for each* $t = (t_0, 0, 0, \cdots) \in \Omega(\hat{a})$, *i.e.,* f *is onto. Let* $(l_2, d^{'})$ *and* $(\mathbb{R}, d)$ *be usual metric spaces. For each* $t, m \in [\frac{\hat{a}_0}{15}, \infty)$, *we obtain*

$$d^{'}(f(t), f(m)) = d^{'}((t, 0, 0, \cdots), (m, 0, 0, \cdots)) = |t - m| = d(t, m).$$

Thus f *is continuous. Additionally,* f^{-1} *is also continuous and bijective, so* f *is homeomorphism. So* $\Omega(\hat{a})$ *is homeomorphic to a closed set* $[\frac{\hat{a}_0}{15}, \infty)$, *hence* $\Omega(\hat{a})$ *is closed. Define retraction* $P_{\Omega(\hat{a})} : \mathcal{B} \to \Omega(\hat{a})$ *by*

$$P_{\Omega(\hat{a})}(l_0, l_1, l_2, \cdots) = \begin{cases} (l_0, l_1, l_2, \cdots), & \text{if } (l_0, l_1, l_2, \cdots) \in \Omega(\hat{a}) \\ (\frac{\hat{a}_0}{15}, 0, 0, \cdots), & \text{if } (l_0, l_1, l_2, \cdots) \notin \Omega(\hat{a}), l_0 < \frac{\hat{a}_0}{15} \\ (l_0, 0, 0, \cdots), & \text{if } (l_0, l_1, l_2, \cdots) \notin \Omega(\hat{a}), l_0 \ge \frac{\hat{a}_0}{15}. \end{cases}$$

In order to demonstrate that $P_{\Omega(\hat{a})}$ *complies with assumption C, we address the subsequent cases.*
Case (a). *For arbitrary* $\hat{a} = \{\hat{a}_n\}, q = \{q_n\}, l = \{l_n\} \in \mathcal{H}$, *assume that* $\hat{a}_0 \le q_0$.

1. *If* $l = \{l_n\} \in \Omega(q)$*, then* $l = \{l_n\} \in \Omega(\hat{a})$ *and hence*

$$\begin{aligned}||P_{\Omega(\hat{a})}(l) - P_{\Omega(q)}(l)||_2 &= ||(l_0, l_1, l_2, \cdots) - (l_0, l_1, l_2, \cdots)||_2 \\ &= 0 \leq \frac{1}{15}||\hat{a} - q||_2.\end{aligned}$$

2. *If* $l = (l_0, l_1, l_2, \cdots) \notin \Omega(q)$ *and* $l = (l_0, l_1, l_2, \cdots) \in \Omega(\hat{a})$*, then either* $l_0 < \frac{q_0}{15}$ *or* $l_0 \geq \frac{q_0}{15}$*. For* $l_0 < \frac{q_0}{15}$*, we have*

$$\begin{aligned}||P_{\Omega(\hat{a})}(l) - P_{\Omega(q)}(l)||_2 &= ||(l_0, l_1, l_2, \cdots) - \Big(\frac{\hat{a}_0}{15}, 0, 0, \cdots\Big)||_2 \\ &= ||(l_0, 0, 0, \cdots) - \Big(\frac{\hat{a}_0}{15}, 0, 0, \cdots\Big)||_2 \\ &= \Big|l_0 - \frac{\hat{a}_0}{15}\Big| = l_0 - \frac{\hat{a}_0}{15} \\ &\leq \frac{1}{15}||\hat{a} - q||_2.\end{aligned}$$

For $l_0 \geq \frac{q_0}{15}$*, we have*

$$\begin{aligned}||P_{\Omega(\hat{a})}(l) - P_{\Omega(q)}(l)||_2 &= ||(l_0, l_1, l_2, \cdots) - (l_0, 0, 0, \cdots)||_2 \\ &= ||(l_0, 0, 0, \cdots) - (l_0, 0, 0, \cdots)||_2 \\ &= 0 \leq \frac{1}{15}||\hat{a} - q||_2.\end{aligned}$$

3. *If* $l = (l_0, l_1, l_2, \cdots) \notin \Omega(q)$ *and* $l = (l_0, l_1, l_2, \cdots) \notin \Omega(\hat{a})$*, then either* $l_0 < \frac{q_0}{15}$ *or* $l_0 \geq \frac{q_0}{15}$ *and* $l_0 < \frac{\hat{a}_0}{15}$ *or* $l_0 \geq \frac{\hat{a}_0}{15}$*. For* $l_0 < \frac{\hat{a}_0}{15}$*, we have* $l_0 < \frac{q_0}{15}$*. Thus,*

$$\begin{aligned}||P_{\Omega(p)}(l) - P_{\Omega(q)}(l)||_2 &= \Big\|\Big(\frac{\hat{a}_0}{15}, 0, 0, \cdots\Big) - \Big(\frac{q_0}{15}, 0, 0, \cdots\Big)\Big\|_2 \\ &= \Big|\frac{\hat{a}_0}{15} - \frac{q_0}{15}\Big| \leq \frac{1}{15}||\hat{a} - q||_2.\end{aligned}$$

For $l_0 \geq \frac{\hat{a}_0}{15}$ *and* $l_0 \geq \frac{q_0}{15}$*, we have*

$$\begin{aligned}||P_{\Omega(\hat{a})}(l) - P_{\Omega(q)}(l)||_2 &= ||(l_0, 0, 0, \cdots) - (l_0, 0, 0, \cdots)||_2 \\ &= 0 \leq \frac{1}{15}||\hat{a} - q||_2.\end{aligned}$$

For $l_0 \geq \frac{\hat{a}_0}{15}$ *and* $l_0 < \frac{q_0}{15}$*, we have*

$$\begin{aligned}||P_{\Omega(p)}(l) - P_{\Omega(q)}(l)||_2 &= ||(l_0, 0, 0, \cdots) - \Big(\frac{q_0}{10}, 0, 0, \cdots)||_2 \\ &= \Big|l_0 - \frac{q_0}{15}\Big| = \frac{q_0}{15} - l_0 \leq \frac{q_0}{15} - \frac{\hat{a}_0}{15} \leq \frac{1}{15}||\hat{a} - q||_2.\end{aligned}$$

Case (b)*. Analogously, for* $\hat{a} = \{\hat{a}_n\}, q = \{q_n\}, r = \{l_n\} \in \mathcal{B}$ *with* $\hat{a}_0 > q_0$*, we can verify that*

$$||P_{\Omega(\hat{a})}(l) - P_{\Omega(q)}(l)||_2 \leq \frac{1}{15}||\hat{a} - q||_2.$$

Thus, $P_{\Omega(\hat{a})}$ *satisfies the assumption C with constant* $\xi = \frac{1}{15}$*. Further, for* $\rho = \frac{1}{2}$ *and* $\varepsilon = 10^{-2}$*, the condition (11) is also satisfied for* $\kappa = \delta = \frac{1}{2}, \tau = \frac{3}{64}, \varsigma = \frac{3}{32}, \sigma = \upsilon = \frac{1}{4}, \theta = \frac{1}{4}, \vartheta = \frac{2}{3}$*. Finally, consider* $\hat{a}^* = (\hat{a}^*_0, 0, 0, \cdots) : \hat{a}^*_0 \geq 0$*, then for* $\hat{a}^* > 0$*, we have*

$$\begin{aligned}\langle \rho(\psi(S(\hat{a}), T(\hat{a})) + \varphi(\hat{a})) - \Psi(u, v), J(\varphi(\hat{b}) - \varphi(\hat{a}))\rangle &= \langle \frac{35}{192}\hat{a}, \frac{1}{2}(\hat{b}) - \frac{1}{2}(\hat{a})\rangle \\ &= \frac{35}{3384}\langle(\hat{a}_0, 0, 0, \cdots), (\hat{b}_0 - \hat{a}_0, 0, 0, \cdots)\rangle \\ &< 0, \forall \hat{b} = (\hat{b}_0, 0, 0, \cdots) \in \Omega(\hat{a}).\end{aligned}$$

However, for $\hat{a} = (0,0,0,\cdots)$*, we have*

$$\begin{aligned}\langle \rho(\psi(S(\hat{a}),T(\hat{a}))+\varphi(\hat{a}))-\Psi(u,v), J(\varphi(\hat{b})-\varphi(\hat{a}))\rangle \\ = \langle (0,0,0,\cdots),(\hat{b}_0-\hat{a}_0,0,0,\cdots)\rangle \\ = 0, \forall \hat{b} = (\hat{b}_0,0,0,\cdots)\in \Omega(\hat{a}).\end{aligned}$$

Hence, $(\hat{a},\mu,v) = (0,0,0)$ *is a unique solution of* $EQVI(\mathcal{B},\psi,\varphi,\Psi)$.

3. ITERATIVE SCHEME AND CONVERGENCE

Next, we discuss the lemma below in Banach spaces plays a pivotal role in establishing the convergence. An identical form proved in Hilbert spaces can be found in [13].

Lemma 6. *Under the suppositions of the Theorem 1, the sequence* $\{\Xi_n\|\hat{a}_n-\hat{a}_{n-1}\|\}$ *converges to 0, where* Ξ_n *is outlined below:*

$$\Xi_n = \begin{cases}\min\Big\{\dfrac{n-1}{n-1+\kappa}, \dfrac{e_n}{\|\hat{a}_n-\hat{a}_{n-1}\|}\Big\}, & \text{if } \hat{a}_n \neq \hat{a}_{n-1},\\ \dfrac{n-1}{n-1+\kappa}, & \text{if } \hat{a}_n = \hat{a}_{n-1},\end{cases} \tag{20}$$

$\forall n\in\mathbb{N}, \kappa \geq 3$ *and* $e_n \in (0,\infty)$ *with* $\lim_{n\to\infty} e_n = 0$.

Proof. The proof can be justified by taking the following possibilities into consideration.
Case (c_1). If $\hat{a}_n = \hat{a}_{n-1}$, then naturally $\{\Xi_n\|\hat{a}_n-\hat{a}_{n-1}\|\}$ is zero sequence. Next, we carry on for $\hat{a}_n \neq \hat{a}_{n-1}$.
Case (c_2). Suppose $\Xi_n = \dfrac{n-1}{n-1+\kappa}, \forall n\in\mathbb{N}$, then
$0 \leq \Xi_n = \dfrac{n-1}{n-1+\kappa} \leq \dfrac{e_n}{\|\hat{a}_n-\hat{a}_{n-1}\|}$ and hence $0 \leq \Xi_n\|\hat{a}_n-\hat{a}_{n-1}\| \leq e_n$.
Case (c_3). Suppose $\Xi_n = \dfrac{e_n}{\|\hat{a}_n-\hat{a}_{n-1}\|}, \forall n\in\mathbb{N}$, then
$0 \leq \Xi_n = \dfrac{e_n}{\|\hat{a}_n-\hat{a}_{n-1}\|} \leq \dfrac{n-1}{n-1+\kappa}$ and hence $0 \leq \Xi_n\|\hat{a}_n-\hat{a}_{n-1}\| = e_n$.
Case (c_4). Finally, suppose $\Xi_n = \dfrac{n-1}{n-1+\kappa} = \dfrac{e_n}{\|\hat{a}_n-\hat{a}_{n-1}\|}$ for some $n\in\mathbb{N}$, then we obtain $0 \leq \Xi_n\|\hat{a}_n-\hat{a}_{n-1}\| = e_n$. Thus, in all cases, $0 \leq \Xi_n\|\hat{a}_n-\hat{a}_{n-1}\| \leq e_n$. Since $\lim_{n\to\infty} e_n = 0$ and hence $\{\Xi_n\|\hat{a}_n-\hat{a}_{n-1}\|\}$ converges to 0.

We reformat the algorithm (8) as follows in light of (10).

Algorithm 1. *Initialization: Given* $\rho > 0$*, initial points* $\hat{a}_0,\hat{a}_1 \in \mathcal{B}, \mu_0 \in A(\hat{a}_0), v_0 \in B(\hat{a}_0)$*. Choose the sequences* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\tau_n\}_{n=1}^{\infty}$ *and* $\{e_n\}_{n=1}^{\infty}$ *so that the assumptions of Lemma 6 hold.*
Step 1: For given iterates $\hat{a}_{n-1},\hat{a}_n$*, choose* Ξ_n *so that*

$$\Xi_n = \begin{cases}\min\Big\{\dfrac{n-1}{n-1+\kappa}, \dfrac{e_n}{\|\hat{a}_n-\hat{a}_{n-1}\|}\Big\}, & \text{if } \hat{a}_n \neq \hat{a}_{n-1},\\ \dfrac{n-1}{n-1+\kappa}, & \text{if } \hat{a}_n = \hat{a}_{n-1}.\end{cases}$$

Step 2: Set

$$\begin{cases}\hat{c}_n = \hat{a}_n + \gamma_n(\hat{a}_n-\hat{a}_{n-1}),\\ \hat{b}_n = \hat{a}_n + \tau_n(\hat{a}_n-\hat{a}_{n-1}).\end{cases}$$

Compute $\zeta_n = (1-\rho)\varphi(\hat{b}_n) - \rho\psi(S(\hat{b}_n),T(\hat{b}_n)) + \Psi(\bar{\mu}_n,\bar{v}_n)] : \bar{\mu}_n \in A(\hat{b}_n), \bar{v}_n) \in B(\hat{b}_n)$.
Step 3: Compute $\hat{a}_{n+1} = (1-\gamma_n-\tau_n)\hat{b}_n + \gamma_n\{\hat{b}_n - \varphi(\hat{b}_n) + P_{\Omega(\hat{b}_n)}\zeta_n\} + \tau_n\hat{c}_n$.
Set $n := n+1$ *and go to Step1.*

Theorem 2. *Suppose that all the hypotheses of Theorem 1 are fulfilled by the mappings* $P_{\Omega(\hat{a})},\varphi,\Psi,\psi,S,T,A$ *and* B. *Suppose that* $\{\hat{a}_n\}_{n=1}^{\infty}$ *is generated by Algorithm 1 with* $\max\{|\gamma_n|,|\tau_n|\} \leq \Xi_n, \forall n\in\mathbb{N}$*, where* Ξ_n *is the updating parameter described in (20). Let* $\{\alpha_n\}_{n=1}^{\infty},\{\beta_n\}_{n=1}^{\infty}$ *and* $\{e_n\}_{n=1}^{\infty}$ *are sequences in* $(0,1)$*, so that* $0 < \alpha_n+\beta_n < 1, \alpha \leq \alpha_n$ *for some* $\alpha > 0$ *and* $\lim_{n\to\infty} e_n = 0, \forall n\in\mathbb{N}$*. Then* $\hat{a}_n \to \hat{a}, \mu_n \to \mu$ *and* $v_n \to v$ *and* $(\hat{a},\mu,v)$ *solves* $EQVI(\mathcal{B},\psi,\varphi,\Psi)$.

Proof. It is evident by the Theorem 1 that $EQVI(\mathcal{B},\psi,\varphi,\Psi)$ admits a solution $(\hat{a},\mu,\nu)$ such that $\hat{a}\in\mathcal{B},\varphi(\hat{a})\in\Omega(\hat{a}),\mu\in A(\hat{a})$ and $\nu\in B(\hat{a})$. Then

$$\hat{a}=\hat{a}-\varphi(\hat{a})+P_{\Omega(\hat{a})}[(1-\rho)\varphi(\hat{a})-\rho\psi(S(\hat{a}),T(\hat{a}))+\Psi(\mu,\nu)]. \tag{21}$$

Next, we substantiate that $\hat{a}_n\to\hat{a}$ an $n\to\infty$. Using the presumptions from Theorem 1, Assumption C, utilizing Lemma 2 and replicating the steps as from (15)-(18), we acquire

$$\begin{aligned}
&||\hat{b}_n-\varphi(\hat{b}_n)+P_{\Omega(\hat{b}_n)}\zeta_n-\hat{a}||\\
&\leq||\hat{b}_n-\hat{a}-(\varphi(\hat{b}_n)-\varphi(\hat{a}))||+\xi||\hat{b}_n-\hat{a}||+\rho||\varphi(\hat{b}_n)-\varphi(\hat{a})||\\
&\quad+||\varphi(\hat{b}_n)-\varphi(\hat{a})-\rho(\psi(S(\hat{b}_n),T(\hat{b}_n))-\psi(S(\hat{a}),T(\hat{a})))||\\
&\quad+||\Psi(\bar{\mu}_n,\bar{\nu}_n)-\Psi(\mu,\nu)||\\
&\leq[\sqrt{1-2\kappa+64\varepsilon\delta^2}+(\xi+\rho\delta)+\sqrt{\delta^2-2\rho\tau+64\varepsilon\varsigma^2}+(\sigma\theta+\upsilon\vartheta)]||\hat{b}_n-\hat{a}||,\\
&=L||\hat{b}_n-\hat{a}||,
\end{aligned} \tag{22}$$

where, L is same as in (19). From Algorithm 1 and (22), we acquire

$$\begin{aligned}
||\hat{a}_{n+1}-\hat{a}||&\leq(1-\alpha_n-\beta_n)||\hat{b}_n-\hat{a}||+\alpha_n||\hat{b}_n-\varphi(\hat{b}_n)\\
&\quad+P_{\Omega(\hat{b}_n)}\zeta_n-\hat{a}||+\beta_n||\hat{c}_n-\hat{a}||\\
&\leq(1-\alpha_n-\beta_n)||\hat{b}_n-\hat{a}||+\alpha_nL||\hat{b}_n-\hat{a}||+\beta_n||\hat{c}_n-\hat{a}||\\
&=(1-\alpha_n-\beta_n+\alpha_nL)||\hat{b}_n-\hat{a}||+\beta_n||\hat{c}_n-\hat{a}||.
\end{aligned} \tag{23}$$

Since $0<L<1$ and $\alpha_n\geq\alpha$, invoking defining properties of $\hat{b}_n,\hat{c}_n$ and (11), (23) turns into

$$\begin{aligned}
||\hat{a}_{n+1}-\hat{a}||&\leq(1-\alpha_n-\beta_n+\alpha_nL)[||\hat{a}_n-\hat{a}||+\tau_n||\hat{a}_n-w_{n-1}||]\\
&\quad+\beta_n[||\hat{a}_n-\hat{a}||+\gamma_n||\hat{a}_n-\hat{a}_{n-1}||]\\
&\leq(1-\alpha_n(1-L)||\hat{a}_n-\hat{a}||+[(1-\alpha_n-\beta_n+\alpha_nL)|\tau_n|\\
&\quad+\beta_n|\gamma_n|]||\hat{a}_n-\hat{a}_{n-1}||\\
&\leq(1-\alpha_n(1-L)||\hat{a}_n-\hat{a}||+[(1-\alpha_n-\beta_n+\alpha_n)|\tau_n|\\
&\quad+\beta_n|\gamma_n|]||\hat{a}_n-\hat{a}_{n-1}||\\
&\leq(1-\alpha(1-L)||\hat{a}_n-\hat{a}||+(|\tau_n|+|\gamma_n|)||\hat{a}_n-\hat{a}_{n-1}||\\
&\leq(1-\alpha(1-L)||\hat{a}_n-\hat{a}||+2\Xi_n||\hat{a}_n-\hat{a}_{n-1}||.
\end{aligned} \tag{24}$$

Invoking the Lemma 6, we acquire $\lim\limits_{n\to\infty}\Xi_n||\hat{a}_n-\hat{a}_{n-1}||=0$ and the assumptions yield $0<(1-\alpha(1-L))<1$. Thus the conclusion $\hat{a}_n\to\hat{a}$ as $n\to\infty$ follows by the Lemma 4. Since $\bar{\mu}_n\in A(\hat{b}_n),\mu\in A(\hat{a})$ and A is θ-$\mathcal{D}$-Lipschitz continuous, then

$$\begin{aligned}
||\bar{\mu}_n-\mu||&\leq\mathcal{D}(A(\hat{b}_n),A(\hat{a}))\leq\theta||\hat{b}_n-\hat{a}||\\
&\leq\theta[||\hat{a}_n-\hat{a}||+\tau_n||\hat{a}_n-\hat{a}_{n-1}||].
\end{aligned} \tag{25}$$

Again from Lemma 6, $\lim\limits_{n\to\infty}\tau_n||\hat{a}_n-\hat{a}_{n-1}||=0$. Hence, $\lim\limits_{n\to\infty}\bar{\mu}_n=\mu$, and in the similar manner, $\lim\limits_{n\to\infty}\bar{\nu}_n=\nu$. Also $\{\hat{c}_n\}$ converges to $\hat{c}$ and $\{\hat{b}_n\}$ converges to $\hat{b}$. Finally, we shall demonstrate that $\mu\in A(\hat{a}),\nu\in B(\hat{a})$.

$$\begin{aligned}
d(\mu,A(\hat{a}))||&\leq||\mu-\mu_n||+d(\mu_n,A(\hat{a}))\\
&\leq||\mu-\mu_n||+\mathcal{D}(A(\hat{a}_n),A(\hat{a}))\\
&\leq||\mu-\mu_n||+\theta||\hat{a}_n-\hat{a}||\to0\ \text{ as } n\to\infty.
\end{aligned} \tag{26}$$

Thus $d(\mu,A(\hat{a}))=0$, thus $\mu\in A(\hat{a})$ and likewise $\nu\in B(\hat{a})$.

4. EXTENDED WEINER-HOPF EQUATION

Now, we construct an extended Wiener-Hopf equation ($EWHE$) and substantiated that it is analogous to $EQVI(\mathcal{B},\psi,\varphi,\Psi)$. By implementing this equivalence, we derive the solution of $EQVI(\mathcal{B},\psi,\varphi,\Psi)$.

We think about $EWHE$ of finding $\{(\hat{z},\hat{a},\mu,\nu):\hat{z},\hat{a}\in\mathcal{B},\mu\in A(\hat{a}),\nu\in B(\hat{a})\}$ so that

$$\varphi(\hat{a})+\rho^{-1}R_{\Omega(\hat{a})}\hat{z}=-\psi(S(\hat{a}),T(\hat{a}))+\rho^{-1}\Psi(\mu,\nu), \tag{27}$$

where $R_{\Omega(\hat{a})}=I_d-P_{\Omega(\hat{a})}, I_d$ is the identity mapping. In the forthcoming theorem, we will correlate the solutions of $EWHE(27)$ and $EQVI(\mathcal{B},\psi,\varphi,\Psi)$.

Theorem 3. *A point $(\hat{z},\hat{a},\mu,\nu)$ such that $\hat{z},\hat{a}\in\mathcal{B},\mu\in A(\hat{a}),\nu\in B(\hat{a})$ solves $EWHE(27)$ if and only if $(\hat{a},\mu,\nu)$ solves $EQVI(\mathcal{B},\psi,\varphi,\Psi)$, where*

$$\varphi(\hat{a})=P_{\Omega(\hat{a})}\hat{z}, \tag{28}$$

$$\hat{z}=(1-\rho)\varphi(\hat{a})-\rho\psi(S(\hat{a}),T(\hat{a}))+\Psi(\mu,\nu). \tag{29}$$

Proof. Assume that $(\hat{z},\hat{a},\mu,\nu)$ such that $\hat{z},\hat{a}\in\mathcal{B},\mu\in A(\hat{a}),\nu\in B(\hat{a})$ is a solution of $EWHE(27)$. Then, one can discern that

$$\rho\varphi(\hat{a})+\rho\psi(S(\hat{a}),T(\hat{a}))-\Psi(\mu,\nu)=-R_{\Omega(\hat{a})}\hat{z}=P_{\Omega(\hat{a})}\hat{z}-\hat{z}. \tag{30}$$

Recalling Lemma 3, (30) turns into

$$\langle P_{\Omega(\hat{a})}\hat{z}-\hat{z},J(\hat{y}-P_{\Omega(\hat{a})}\hat{z})\rangle\geq 0 \tag{31}$$

which yields,

$$\langle\rho(\varphi(\hat{a})+\psi(S(\hat{a}),T(\hat{a})))-\Psi(\mu,\nu),J(\hat{y},P_{\Omega(\hat{a})}\hat{z})\rangle\geq 0.$$

Thus, $\varphi(\hat{a})=P_{\Omega(\hat{a})}[(1-\rho)\varphi(\hat{a})-\rho\psi(S(\hat{a}),T(\hat{a}))+\Psi(\mu,\nu)]$, and hence, $EQVI(\mathcal{B},\psi,\varphi,\Psi)$ admits a solution $(\hat{a},\mu,\nu)$. Conversely, suppose $(\hat{a},\mu,\nu)$ solves $EQVI(\mathcal{B},\psi,\varphi,\Psi)$, then

$$\varphi(\hat{a})=P_{\Omega(\hat{a})}[(1-\rho)\varphi(\hat{a})-\rho\psi(S(\hat{a}),T(\hat{a}))+\Psi(\mu,\nu)]. \tag{32}$$

Utilizing the fact $R_{\Omega(\hat{a})}=I_d-P_{\Omega(\hat{a})}$ and (32), we get

$$\begin{aligned}&R_{\Omega(\hat{a})}[(1-\rho)\varphi(\hat{a})-\rho\psi(S(\hat{a}),T(\hat{a}))+\Psi(\mu,\nu)]\\&=(1-\rho)\varphi(\hat{a})-\rho\psi(S(\hat{a}),T(\hat{a}))+\Psi(\mu,\nu)\\&\quad-P_{\Omega(\hat{a})}[(1-\rho)\varphi(\hat{a})-\rho\psi(S(\hat{a}),T(\hat{a}))+\Psi(\mu,\nu)].\end{aligned} \tag{33}$$

Using (28)-(30), (33) turns into

$$\varphi(\hat{a})+\rho^{-1}R_{\Omega(\hat{a})}\hat{z}=-\psi(S(\hat{a}),T(\hat{a}))+\rho^{-1}\Psi(\mu,\nu).$$

Thus, $(\hat{z},\hat{a},\mu,\nu)$ is a solution of $EWHE(27)$.

Now, we shall put forward the following iteration process by taking into consideration (28)-(29) and recalling the Nadlar's technique [36].

Algorithm 2. *For initial points $\hat{a}_0,\hat{z}_0\in\mathcal{B},\mu_0\in A(\hat{a}_0),\nu_0\in B(\hat{a}_0)$, we estimate $\{\hat{a}_n\},\{\hat{z}_n\},\{\mu_n\},\{\nu_n\}$ as under:*

$$\begin{aligned}&\varphi(\hat{a}_n)=P_{\Omega(\hat{a}_n)}\hat{z}_n,\\&\hat{z}_{n+1}=(1-\rho)\varphi(\hat{a}_n)-\rho\psi(S(\hat{a}_n),T(\hat{a}_n))+\Psi(\mu_n,\nu_n),\\&\mu_n\in A(\hat{a}_n):\|\mu_{n+1}-\mu_n\|\leq\mathcal{D}(A(\hat{a}_{n+1}),A(\hat{a}_n)+\pi^{n+1}\|\hat{a}_{n+1}-\hat{a}_n\|\\&\nu_n\in B(\hat{a}_n):\|\nu_{n+1}-\nu_n\|\leq\mathcal{D}(B(\hat{a}_{n+1}),B(\hat{a}_n))+\pi^{n+1}\|\hat{a}_{n+1}-\hat{a}_n\|.\end{aligned} \tag{34}$$

Theorem 4. *Suppose that $\varphi:\mathcal{B}\to\mathcal{B}$ be k-expanded, δ-Lipschitz continuous mapping; $\Psi,\psi:\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ and $S,T:\mathcal{B}\to\mathcal{B}$ be the single-valued mappings so that $\psi(S,T)$ is τ-strongly accretive regarding φ and ς-Lipschitz continuous, Ψ is (σ,υ)-mixed Lipschitz continuous; $A,B:\mathcal{B}\to CB(\mathcal{B})$ be θ-$\mathcal{D}$-Lipschitz continuous, ϑ-$\mathcal{D}$-Lipschitz continuous, respectively. If the retraction $P_{\Omega(\hat{a})}:\mathcal{B}\to\Omega(\hat{a})$ comply with the Assumption C and constant $\rho>0$ satisfies:*

$$\rho\delta+\sqrt{\delta^2-2\rho\tau+64\varepsilon\varsigma^2}+(\sigma\theta+\upsilon\vartheta)<k-\xi,\tau<\frac{\delta^2+64\varepsilon\varsigma^2}{2\rho}. \tag{35}$$

Then the iterative sequences $\{\hat{z}_n\},\{\hat{a}_n\},\{\mu_n\},\{\nu_n\}$ produced by Algorithm 2 converges to $\hat{z},\hat{a},\mu_n,\nu_n$, respectively, and $(\hat{z},\hat{a},\mu_n,\nu_n)$ so that $\hat{z},\hat{a}\in\mathcal{B},\mu_n\in A(\hat{a}),\nu_n\in B(\hat{a})$ solves $EWHE(27)$.

Proof. By implementing scheme (34), we get

$$\begin{aligned}\|\hat{z}_{n+2}-\hat{z}_{n+1}\| &= \|(1-\rho)\varphi(\hat{a}_{n+1})-\rho\psi(S(\hat{a}_{n+1}),T(\hat{a}_{n+1}))+\Psi(\mu_{n+1},\nu_{n+1})\\ &\quad -[(1-\rho)\varphi(\hat{a}_n)-\rho\psi(S(\hat{a}_n),T(\hat{a}_n))+\Psi(\mu_n,\nu_n)]\\ &\leq \rho\|\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)\|+\|\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)-\rho(\psi(S(\hat{a}_{n+1}),\\ &\quad -T(\hat{a}_{n+1}))\psi(S(\hat{a}_n),T(\hat{a}_n)))\|+\|\Psi(\mu_{n+1},\nu_{n+1})-\Psi(\mu_n,\nu_n)\|.\end{aligned} \tag{36}$$

Since φ is δ-Lipschitz continuous, $\psi(S,T)$ is τ-strongly accretive regarding φ and ς-Lipschitz continuous, then

$$\begin{aligned}&\|\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)-\rho(\psi(S(\hat{a}_{n+1}),T(\hat{a}_{n+1}))-\psi(S(\hat{a}_n),T(\hat{a}_n)))\|^2\\ &\leq \|\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)\|^2-2\rho\langle\psi(S(\hat{a}_{n+1}),T(\hat{a}_{n+1}))-\psi(S(\hat{a}_n),T(\hat{a}_n))),\\ &\quad J(\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)-\rho(\psi(S(\hat{a}_{n+1}),T(\hat{a}_{n+1}))-\psi(S(\hat{a}_n),T(\hat{a}_n))))\rangle\\ &\leq \|\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)\|^2-2\rho\langle\psi(S(\hat{a}_{n+1}),T(\hat{a}_{n+1}))-\psi(S(\hat{a}_n),T(\hat{a}_n))),\\ &\quad J(\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n))\rangle+2\rho\langle\psi(S(\hat{a}_{n+1}),T(\hat{a}_{n+1}))-\psi(S(\hat{a}_n),T(\hat{a}_n))),\\ &\quad J(\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)-\rho(\psi(S(\hat{a}_{n+1}),T(\hat{a}_{n+1}))-\psi(S(\hat{a}_n),T(\hat{a}_n)))\\ &\quad -J(\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n))\rangle\\ &\leq \delta^2\|\hat{a}_{n+1}-\hat{a}_n\|^2-2\rho\tau\|\hat{a}_{n+1}-\hat{a}_n\|^2+64\varepsilon\theta^{\|}\hat{a}_{n+1}-\hat{a}_n\|^2\\ &= (\delta^2-2\rho\tau+64\varepsilon\varsigma^2)\|\hat{a}-\hat{b}\|^2.\end{aligned} \tag{37}$$

Employing (σ,υ)-mixed Lipschitz continuity of Ψ, and Lipschitz continuities of A and B yields

$$\begin{aligned}&\|\Psi(\mu_{n+1},\nu_{n+1})-\Psi(\mu_n,\nu_n)\|\\ &\leq \sigma\|\mu_{n+1}-\mu_n\|+\upsilon\|\nu_{n+1}-\nu_n\|\\ &\leq \sigma\mathscr{D}(A(\hat{a})_{n+1}),A(\hat{a}_n))+\upsilon\mathscr{D}(B(\hat{a}_{n+1}),B(\hat{a}_n))\\ &\leq \sigma(\theta+\pi^{n+1})\|\hat{a}_{n+1}-\hat{a}_n\|+\upsilon(\vartheta+\pi^{n+1})\|\hat{a}_{n+1}-\hat{a}_n\|\\ &= [\sigma(\theta+\pi^{n+1})+\upsilon(\vartheta+\pi^{n+1})]\|\hat{a}_{n+1}-\hat{a}_n\|.\end{aligned} \tag{38}$$

Combining (37) and (38), we obtain

$$\|\hat{z}_{n+2}-\hat{z}_{n+1}\| \leq [\rho\delta+\sqrt{\delta^2-2\rho\tau+64\varepsilon\varsigma^2}+\sigma(\theta+\pi^{n+1})+\upsilon(\vartheta+\pi^{n+1})]\|\hat{a}_{n+1}-\hat{a}_n\|. \tag{39}$$

Since φ is k-expanded, then

$$\begin{aligned}k\|\hat{a}_{n+1}-\hat{a}_n\| &\leq \|\varphi(\hat{a}_{n+1})-\varphi(\hat{a}_n)\|\\ &= \|P_{\Omega(\hat{a}_{n+1})}\hat{z}_{n+1}-P_{\Omega(\hat{a}_n)}\hat{z}_n\|\\ &\leq [\|\hat{z}_{n+1}-\hat{z}_n\|+\xi\|\hat{a}_{n+1}-\hat{a}_n\|],\end{aligned} \tag{40}$$

that results in

$$\|\hat{a}_{n+1}-\hat{a}_n\| \leq \frac{1}{k-\xi}\|\hat{z}_{n+1}-\hat{z}_n\|. \tag{41}$$

Making use of (41) into (39), we obtain

$$\|\hat{z}_{n+2}-\hat{z}_{n+1}\| \leq \Gamma_n\|\hat{z}_{n+1}-\hat{z}_n\|, \tag{42}$$

where $\Gamma_n = \dfrac{[\rho\delta+\sqrt{\delta^2-2\rho\tau+64\varepsilon\varsigma^2}+\sigma(\theta+\pi^{n+1})+\upsilon(\vartheta+\pi^{n+1})]}{k-\xi}$. Indeed, the hypothesis $\pi\in(0,1)$, results in $\lim_{n\to\infty}\Gamma_n=\Gamma$, where $\Gamma = \dfrac{[\rho\delta+\sqrt{\delta^2-2\rho\tau+64\varepsilon\varsigma^2}+(\sigma\theta+\upsilon\vartheta)]}{k-\xi}$. By (35), $\Gamma<1$ and hence $\{\hat{z}_n\}$ is a Cauchy sequence in $\mathscr{B}$. Then, we can find $\hat{z}\in\mathscr{B}$ so that $\lim_{n\to\infty}\hat{z}_n=\hat{z}$. We can therefore conclude from (41), that $\{\hat{a}_n\}$ is a Cauchy sequence in $\mathscr{B}$ as well. One can find $\hat{a}\in\mathscr{B}$ so that $\lim_{n\to\infty}\hat{a}_n=\hat{a}$. Invoking A's Lipschitz continuity, we manifest that

$$\begin{aligned}\|\mu_{n+1}-\mu_n\| &\leq \mathscr{D}(A(\hat{a}_{n+1}),A(\hat{a}_n))+\pi^{n+1}\|\hat{a}_{n+1})-\hat{a}_n\|\\ &\leq (\theta+\pi^{n+1})\|\hat{a}_{n+1}-\hat{a}_n\|.\end{aligned} \tag{43}$$

Since the sequence $\{\hat{a}_n\}$ is Cauchy in $\mathcal{B}$ and so is $\{\mu_n\}$. In the same manner, we manifest that $\{\nu_n\}$ is Cauchy in $\mathcal{B}$. Hence $\lim_{n\to\infty} \mu_n = \mu$ and $\lim_{n\to\infty} \nu_n = \nu$ for some $\mu, \nu \in \mathcal{B}$. Since $\mu_n \in A(\hat{a}_n)$, then

$$\begin{aligned} d(\mu, A(\hat{a}))|| &\le ||\mu - \mu_n|| + d(\mu_n, A(\hat{a})) \\ &\le ||\mu - \mu_n|| + \mathcal{D}(A(\hat{a}_n), A(\hat{a})) \\ &\le ||\mu - \mu_n|| + \theta||\hat{a}_n - \hat{a}|| + \pi^n||\hat{a}_n - \hat{a}|| \\ &= ||\mu - \mu_n|| + (\theta + \pi^n)||\hat{a}_n - \hat{a}|| \to 0 \ \text{ as } n \to \infty. \end{aligned} \tag{44}$$

Thus, $d(\mu, A(\hat{a})) = 0$ and hence $\mu \in A(\hat{a})$ and correspondingly, $\nu \in B(\hat{a})$. Consequently, we draw the conclusion that $\hat{a}_n \to \hat{a}, \hat{z}_n \to \hat{z}, \mu_n \to \mu$ and $\nu_n \to \nu$ and the continuity of the mappings $P_{\Omega(\hat{a})}, \varphi, \psi, S, T, \Psi, A, B$, we achieve $\varphi(\hat{a}) = P_{\Omega(\hat{a})}\hat{z}$, where, $\hat{z} = (1-\rho)\varphi(\hat{a}) - \rho\psi(S(\hat{a}), T(\hat{a})) + \Psi(\mu, \nu)$, which amounts to say $(\hat{z}, \hat{a}, \mu, \nu)$ is a solution of $EWHE(27)$.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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