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Exploring Advanced Analysis Technique for Shallow Water Flow Models with Diverse Applications

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ABSTRACT: The present article focuses on the analytical approach using fractional orders and its application in the dynamics of physical processes. Fractional order models align better with experimental data compared to non-fractional ones. This study primarily focuses on employing the new approximate analytical method to solve shallow water models with fractional orders. Numerical examples within the Caputo fractional derivative showcase the method's application. Results for both integer and fractional orders are graphically depicted, demonstrating the fractional solutions' closeness to actual data. Analysis of 3D and 2D fractional order graphs reveals convergence toward integer order graphs as fractional derivatives approach non-fractional ones. This method shows promise for direct application in solving targeted problems and can be easily adapted for other fractional nature problems.

Keywords: Water flow models, fractional view analysis, analytical solution, new approximate analytical method, Riemann-Liouuille partial fractional order operator of integration, Caputo operator.

1. INTRODUCTION

Classical calculus is a fundamental subject in applied mathematics, transforming complex physical processes into simple mathematical framework. This mathematical framework represents the simplest expression of all physical phenomena, making them easier to understand and manipulate. However, the rapid advancement of science and technology has shifted researchers' attention towards modeling complex phenomena that are not adequately addressed by classical calculus. Consequently, researchers have redirected their focus to fractional calculus.

Fractional Calculus (FC) has attracted researchers because of its broad scope and applications in various branches of science and technology. Some of these new applications of FC can be seen in modeling numerous physical processes, such as the lung parenchyma stress relaxation model using fractional viscoelasticity [1], one-dimensional blood flow model using fractional-order viscoelasticity [2], and the modeling of epoxy resin with fractional order [3]. Additional applications include capturing the non-linear hereditariness of tendons and ligaments in the human knee using non-integer orders [4], computer simulations of real-case structures [5], viscoelastic analysis of hereditary-aging structures with fractional order [6], analyzing the dynamic behavior of fractional order cancer models [7], parallel RCL circuit models with Caputo-Fabrizio fractional order derivatives [8], fractional-order Langevin equations driven by periodically modulated noise with mass fluctuation [9], and Atangana–Baleanu fractional models for Jeffrey's flow nanofluid [10].

Due to the accurate modeling of various physical phenomena using FC, researchers have shown great interest in developing numerical and analytical techniques for the solution of these models. In this connection, mathematicians have extended and applied well-known techniques such as the homotopy analysis method [11], sine-cosine method [12], homotopy perturbation method [13], homotopy perturbation transform method [14], transform method [15], reproducing kernel method [16], exponential rational function method [17], improved F-expansion method [18], modified simple equation method [19], first integral method [20], auxiliary equation method [21], Kudryashov method [22], Laplace Adomian decomposition method [23], Mohand transform [24], local and global meshless methods [25–29], new analytical techniques [30], modified decomposition method [31], Hamiltonian approach [32], and variational iteration algorithms and their modifications [33–38]. In this case study, a new approximate analytical method (NAAM) is implemented to investigate the solutions of fractional flow of liquids' models (shallow water wave models with fractional order). The proposed technique is based on Riemann-Caputo operator (23), its basic properties and a simple decomposition procedure. For application part, the fractional fluid flow models are considered to confirming the validity of the suggested method.
The first fluid flow model is shallow water model in the form:

$$\zeta_\tau^{\wp}(r,\tau)+\pounds\zeta_r(r,\tau)+\zeta(r,\tau)\zeta_r(r,\tau)+\zeta_{rr\tau}(r,\tau)-\hbar\zeta_r(r,\tau)\zeta_{rr}(r,\tau)-\hbar\zeta(r,\tau)\zeta_{rr\tau}(r,\tau),\quad \wp\in(0,1] \tag{1}$$

with initial source

$$\zeta(r,0)=\psi(r).$$

The shallow water flow model is named Burger Poisson Equation and usually used to define many physical phenomena in viscous fluid such as shallow water flow and shock waves.
The second fluid flow model within fractional derivative can be written as

$$\zeta_\tau^{\wp}(r,\tau)-\zeta_{rr\tau}(r,\tau)+\pounds\zeta(r,\tau)=\pounds\zeta(r,\tau)\zeta_{rr\tau}(r,\tau)-\zeta(r,\tau)\zeta_(r)+\hbar\zeta_r(r,\tau)\zeta_{rr},\quad \wp\in(0,1] \tag{2}$$

with initial source

$$\zeta(r,0)=\delta(r).$$

It is designed for multi-directional nonlinear dispersive wave in shallow waters and is known as Fornberg–Whitham equation of time fractional order.
The third fluid flow model with derivative of fractional order can be expressed as

$$\begin{aligned}\zeta_\tau^{\wp}(r,\tau)+\zeta(r,\tau)\zeta_r(r,\tau)+\xi_r(r,\tau)+b\zeta_{rr}(r,\tau)=0,\\ \xi_\tau^{\wp}(r,\tau)+\zeta(r,\tau)\zeta_r(r,\tau)+a\xi_{rrr}(r,\tau)-b\xi_{rr}(r,\tau)=0,\end{aligned}\quad \wp\in(0,1] \tag{3}$$

with initial sources,

$$\zeta(r,0)=\varphi(r).$$

$$\xi(r,0)=\psi(r).$$

It is the system used for shallow water phenomena and known as non-linear Whitham-Broer-Kaup fractional order model. The aforementioned models are solved using NAAM, known for its straightforward approach. The results obtained exhibit rapid convergence towards the exact solutions for each problem. Additionally, graphical analyses comparing the obtained and exact solutions demonstrate close correspondence. The fractional order 2-D and 3-D graphical analysis are done for application purpose.
The article is structured as follows: Section (2) introduces basic definitions, Section (3) details the implementation of the New Approximate Analytical Method, Section (4) showcases NAAM applications in solving fluid flow models, Section (5) concludes the results and discussions regarding NAAM in fluid flow models, and finally, Section (6) presents the conclusion.

2. BASIC PRELIMINARIES RESULTS

In this section, we presented the NAM procedures along with the basic definitions and results used for completion of this research article.

2.1 RIEMANN-LIOUUILLE OPERATOR OF INTEGRATION

The Riemann-Liouuille integral of partial fractional order is represented by is $I_\tau^\wp$, where, $\wp \in N, \wp \geq 0$, and is defined as under.

$$I_\tau^\wp \xi(r,\tau) = \begin{cases} \frac{1}{r(\wp)} \int_0^\tau \xi(r,\tau) d\tau, & \wp, \tau > 0, \\ \xi(r,\tau), & \wp = 0, \tau > 0, \end{cases} \tag{4}$$

where, r represent the gamma function.

2.2 SOME FUNDAMENTAL PROPERTIES OF RIEMANN-LIOUUILLE INTEGRAL

ILet $\wp, \theta \in R \setminus N$, $\wp, \theta > 0, \varrho > -1$, then the function $\xi(r,\tau)$ with respect to integration $I_\tau^\wp$, we have

$$\begin{cases} I_\tau^\wp \xi(r,\tau) I_\tau^\theta \xi(r,\tau) = I_\tau^{\wp+\theta} \xi(r,\tau), \\ I_\tau^\wp \xi(r,\tau) I_\tau^\theta \xi(r,\tau) = I_\tau^\theta \xi(r,\tau) I_\tau^\wp \xi(r,\tau), \\ I_\tau^\wp \tau^\varrho = \frac{r(\rho+1)}{r(\wp+\varrho+1)} \tau^{\wp+\varrho}. \end{cases} \tag{5}$$

2.3 BASIC DEFINITION OF CAPUTO OPERATOR [?]

$$D_\tau^\wp \xi(r,\tau) = \frac{\partial^\wp \xi(r,\tau)}{\partial \tau^\wp} = \begin{cases} I^{n-\wp}[\frac{\partial^\wp \xi(r,\tau)}{\partial \tau^\wp}], & n-1 < \wp < n, \quad n \in N \\ \frac{\partial^\wp \xi(r,\tau)}{\partial \tau^\wp}, & n = \wp \end{cases} \tag{6}$$

2.3.1. *Some combine properties of Riemann and Caputo operator*

Let $\wp, \tau \in R$, $\tau > 0$, and $\aleph- < \rho < \aleph \in N$, then

$$\begin{aligned} & I_\tau^\wp D_t^\wp \xi(r,\tau) = \xi(r,\tau) - \sum_{k=0}^{\aleph-1} \frac{\tau^k}{k!} \frac{\partial^k \xi(r,0^+)}{\partial \tau^k}. \\ & D_\tau^\wp I_\tau^\wp \xi(r,\tau) = \xi(r,\tau) \end{aligned} \tag{7}$$

3. THE GENERALIZE NAAM IMPLEMENTATION FOR THE SOLUTION OF WATER WAVE MODELS [39]

n this section, we elaborated the NAAM general procedure for the analytical solution of water flow models. We have Considered general time fractional non-linear fluid flow model in the form;

$$\zeta_\tau^\wp(r,\tau) = £\zeta(r,\tau) + \aleph\zeta(r,\tau) + g(r,\tau), \quad \wp \in [1,2] \tag{8}$$

with initial source

$$\zeta(r,0) = \zeta(r),$$

where £ is taken as a linear operator, while $\aleph$ is non-linear operator.
For the implementation of NAAM, we expressed the required definition and some basic results which required for the computational procedure as follows.

3.1 LEMMA [30]

For $\zeta(r,\tau) = \sum_0^\infty \rho^k \zeta_\kappa(r,\tau)$, the linear operator $£\zeta(r,\tau)$ has the following satisfied property.

$$£\zeta(r,\tau) = £\left(\sum_{\kappa=0}^\infty \rho^\kappa \zeta_\kappa(r,\tau)\right) = \sum_{\kappa=0}^\infty £\left(\rho^\kappa \zeta_\kappa(r,\tau)\right) \tag{9}$$

3.2 THEOREM [30]

et $\zeta(r,\tau) = \sum_0^\infty \zeta_\kappa(r,\tau)$, for the parameter λ, we define $\zeta_\lambda(r,\tau) = \sum_0^\infty \lambda^\kappa \zeta_\kappa(r,\tau)$, then the nonlinear operator $\aleph\zeta(r,\tau)$, satisfy the following property.

$$\aleph(\zeta_\lambda) = \aleph\left(\sum_0^\infty \lambda^\kappa \zeta_\kappa(r,\tau)\right) = \sum_0^\infty \left[\frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\aleph\left(\sum_0^\infty \lambda^\kappa \zeta_\kappa(r,\tau)\right)\right]_{\lambda=0}\right] \lambda^n. \tag{10}$$

3.3 DEFINITION [1]

The polynomial $P_n = P_n(\zeta_0, \zeta_1, \cdots, \zeta_n)$, can be calculated as

$$P_n(\zeta_0, \zeta_1, \cdots, \zeta_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\aleph\left(\sum_0^{\infty} \lambda^{\kappa}\zeta_{\kappa}(r,\tau)\right)\right]_{\lambda=0}. \tag{11}$$

3.4 DEFINITION [30]

If the $P_n = P_n(\zeta_0, \zeta_1, \cdots, \zeta_n)$, by using definition (11), the nonlinear operator $\aleph(\zeta_\lambda)$ is express as

$$\aleph(\zeta_\lambda) = \sum_0^{\infty} \lambda^{\kappa} P_{\kappa}. \tag{12}$$

3.5 VERIFICATION OF NAAM SOLUTION (UNIQUENESS AND EXISTENCE) [39]

The uniqueness and existence of NAAM are described briefly by the stated theorems.

3.5.1. Theorem

Let $g(r,\tau), \zeta(r,\tau)$ are define for $m-1 < \wp < m$, in (8). Then the water flow model (8), gives the unique solution as

$$\zeta(r,\tau) = g_{\tau}^{-\wp}(r,\tau) + \zeta(r) + \sum_{\kappa=1}^{\infty}\left[\pounds_{\tau}^{-\wp}(\zeta_{(\kappa-1)}) + P_{(\kappa-1)\tau}^{-\wp}\right], \tag{13}$$

where, $\pounds_{\tau}^{-\wp}(\zeta_{(\kappa-1)})$ and $P_{(\kappa-1)\tau}^{-\wp}$ represent the fractional partial integral of order $\wp$ for $\pounds(\zeta_{\kappa-1})$ and $P_{(\kappa-1)}$ with respect to τ.
Proof: Ruminate the solution of water model $\zeta(r,\tau)$ is obtained by using the following expansion.

$$\zeta(r,\tau) = \sum_{\kappa=0}^{\infty} \zeta_{\kappa}(r,\tau). \tag{14}$$

The solution of equation (8), along with initial condition can be analyzed as

$$\zeta_{\tau\lambda}^{\wp}(r,\tau) = \lambda[\pounds\zeta(r,\tau) + \aleph\zeta(r,\tau) + g(r,\tau)], \quad \lambda \in [0,1] \tag{15}$$

with initial source

$$\zeta(r,0) = \zeta(r). \tag{16}$$

Furthermore, the solution of equation (13) is approximated as

$$\zeta_{\lambda}(r,\tau) = \sum_0^{\infty} \lambda^{\kappa}\zeta(r,\tau). \tag{17}$$

Applying the Riemann-Liouuille operator of order $\wp$ with respect to τ on both side of equation (15) with the combination of the property of Riemann-Liouuille fractional operator (23), we have

$$\zeta_{\lambda}(r,\tau) = \zeta(r,0) + \lambda I_{\tau}^{\wp}\left[\pounds\zeta(r,\tau) + \aleph\zeta(r,\tau) + g(r,\tau)\right], \tag{18}$$

using equation (16), and initial source, equation (18), can be written as

$$\zeta_{\lambda}(r,\tau) = \zeta(r) + \lambda I_{\tau}^{\wp}\left[\pounds\zeta(r,\tau) + \aleph\zeta(r,\tau) + g(r,\tau)\right]. \tag{19}$$

Using equations (17) and (19), we get

$$\sum_{\kappa=0}^{\infty}\lambda^{\kappa}\zeta_{\lambda}(r,\tau) = \zeta(r) + \lambda\left[g(r,\tau)\right] + \lambda I_{\tau}^{\wp}\left[\pounds\left(\sum_{\kappa=0}^{\infty}\lambda^{\kappa}\zeta(r,\tau)\right) + \aleph\left(\sum_{\kappa=0}^{\infty}\lambda^{\kappa}\zeta(r,\tau)\right)\right], \tag{20}$$

with the help of lemma (3.1) and theorem (3.4), equation (20), become

$$\sum_{\kappa=0}^{\infty}\lambda^{\kappa}\zeta_{\lambda}(r,\tau) = \zeta(r) + \lambda\left[g(r,\tau)\right] + \lambda I_{\tau}^{\wp}\left[\pounds\left(\sum_{\kappa=0}^{\infty}\lambda^{\kappa}\lambda^{\kappa}\zeta_{\kappa}(r,\tau)\right)\right] + \lambda I_{\tau}^{\wp}\left[\aleph\left(\sum_{\kappa=0}^{\infty}\lambda^{\kappa}P_n\right)\right], \tag{21}$$

in equation (21), we equated the coefficients of the same powers of λ, we get the iterative scheme as

$$\begin{cases} \zeta_0(r,\tau) = \zeta(r) \\ \zeta_1(r,\tau) = g^{-\wp}(r,\tau) + \pounds_{\tau}^{-\wp}\zeta_0(r,\tau) + P_{\tau 0}^{-\wp} \\ \zeta_{\kappa}(r,\tau) = \pounds_{\tau}^{-\wp}\zeta_{(\kappa-1)}(r,\tau) + P_{\tau(\kappa-1)}^{-\wp}, \quad \kappa = 2, 3, \cdots \end{cases} \tag{22}$$

4. APPLICATIONS AND RESULTS OF NAAM [40]

In this section, the analytical solutions of the three fractional fluid flow models by using the NAAM. The graphical representation will be presented to discuss the obtained results in an elaborated way.

Model 4.1: We have consider the shallow water flow model (Burger Poisson Equation) as

$$\zeta_{\tau}^{\wp}(r,\tau) + \pounds + \zeta_r(r,\tau) + \zeta(r,\tau)\zeta_r(r,\tau) + \zeta_{rr\tau}(r,\tau) - \hbar\zeta_r(r,\tau)\zeta_{rr}(r,\tau) - \hbar\zeta(r,\tau)\zeta_{rr\tau}(r,\tau), \quad \wp \in (0,1] \tag{23}$$

with initial source

$$\zeta(r,0) = 1 - r,$$

at $\wp = 1, \pounds = 1, \hbar = 3$, the analytical solution is

$$\zeta(r,\tau) = \frac{-r+\tau}{1-\tau}.$$

To solve the FBP equation (23), compare it with the equation (8), we get

$$\zeta_{\tau}^{\wp}(r,\tau) = -\pounds\zeta_r(r,\tau) - \zeta(r,\tau)\zeta_r(r,\tau) - \zeta_{rr\tau}(r,\tau) + \hbar\zeta_r(r,\tau)\zeta_{rr}(r,\tau) + \hbar\zeta(r,\tau)\zeta_{rr\tau}(r,\tau), \quad \wp \in (0,1], \tag{24}$$

The assumed solution of equation (23), is taken as

$$\zeta(r,\tau) = \sum_{\kappa=0}^{\infty} \zeta_{\kappa}(0,\tau). \tag{25}$$

The analytical procedure of equation (24), can be attained through the following procedure

$$\frac{\partial^{\wp}\zeta(r,\tau)}{\partial t^{\wp}} = \lambda\left[-\pounds\zeta_r(r,\tau) + \zeta_{rr\tau}(r,\tau) + \aleph(\zeta(r,\tau))\right], \quad \wp \in (0,1], \tag{26}$$

where the non-linear term $\aleph(\zeta) = -\zeta(r,\tau)\zeta_r(r,\tau) + \hbar\zeta_r(r,\tau)\zeta_{rr}(r,\tau) + \hbar\zeta(r,\tau)\zeta_{rr\tau}(r,\tau)$, and initial source is given as

$$\zeta(r,0) = 1 - r. \tag{27}$$

For (26), the assume solution is in the form

$$\zeta_{\lambda}(r,\tau) = \sum_{\kappa=0}^{\infty} \lambda^{\kappa}\zeta_{\kappa}(r,\tau). \tag{28}$$

Applying the Riemann-Liouuille operator fractional order $\wp$ of integration with independent variable τ on both side of equation (26), along with property (5), and initial source (27), we obtain

$$\zeta_{\lambda}(r,\tau) = \zeta(r,0) + \lambda I_{\tau}^{\wp}\left[-\pounds\zeta_r(r,\tau) + \zeta_{rr\tau}(r,\tau) + \aleph(\zeta(r,\tau))\right], \tag{29}$$

with the help of definition (12), and equation (58), we obtain

$$\sum_{\kappa=0}^{\infty} \lambda^{\kappa}\zeta_{\kappa}(r,\tau) = \zeta(r,0) + \lambda I_{\tau}^{\wp}\left[\sum_{\kappa=0}^{\infty} \lambda^{\kappa}(-\pounds\zeta_r(r,\tau)) + \sum_{\kappa=0}^{\infty} \lambda^{\kappa}(\zeta_{rr\tau}(r,\tau)) + \sum_{\kappa=0}^{\infty} \lambda^{\kappa}P_{\kappa}\right], \tag{30}$$

by equating those terms having same identical power of λ of equation (30), we obtain initial components and the recursive scheme in the form

$$\begin{cases} \zeta_0(r,\tau) = \zeta(r,0), \\ \zeta_1(r,\tau) = I_{\tau}^{\wp}\left[(-\zeta_{0r}(r,\tau)) + (\zeta_{0rr\tau}(r,\tau)) + P_0\right], \\ \zeta_{\kappa}(\chi,r,\tau) = I_{\tau}^{\wp}\left[-\pounds\zeta_{(\kappa-1)r}(r,\tau) + \zeta_{(\kappa-1)rr\tau}(r,\tau) + P_{(\kappa-1)}\right]. \end{cases} \tag{31}$$

Thus evaluating for $\hbar = 3, \pounds = 1$, and simplifying, we obtain

$$\zeta_0(r,\tau) = -r, \tag{32}$$

$$\zeta_1(r,\tau) = (1-r)\frac{\tau^{\wp}}{r(\wp+1)}, \tag{33}$$

$$\zeta_2(r,\tau) = 2(1-r)\frac{\tau^{2\wp}}{r(2\wp+1)}, \tag{34}$$

$$\zeta_3(r,\tau) = 4(1-r)\frac{\tau^{2\wp+1}}{r(2\wp+2)} + r(2\wp)(1-r)\frac{\tau^{3\wp}}{r(2\wp+2)r(3\wp)(r(\wp))^2} \tag{35}$$

$$\vdots$$

The NAAM solution is

$$\zeta(r,\tau) = \zeta_0(r,\tau) + \zeta_1(r,\tau) + \zeta_2(r,\tau) + \zeta_3(r,\tau) + \cdots \tag{36}$$

Substituting, equations (32,33,34,35), in equation (36), we get

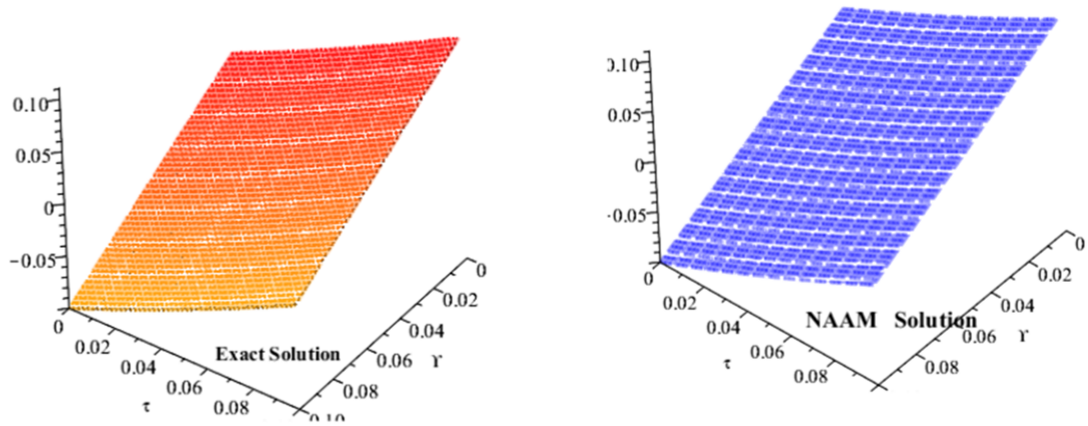
$$\zeta(r,\tau) = -r + (1-r)\frac{\tau^{\wp}}{r(\wp+1)} + 2(1-r)\frac{\tau^{2\wp}}{r(2\wp+1)} + 4(1-r)\frac{\tau^{2\wp+1}}{r(2\wp+2)} + r(2\wp)(1-r)\frac{\tau^{3\wp}}{r(2\wp+2)r(3\wp)(r(\wp))^2} + \cdots \tag{37}$$

Specifically for $\wp = 1$, the solution become as

$$\zeta(r,\tau) = -r + (1-r)\tau + (1-r)\tau^2 + (1-r)\tau^3 + \cdots \tag{38}$$

The analytical solution gradually approach the exact solution as

$$\zeta(r,\tau) = \frac{-r+\tau}{1-\tau} \tag{39}$$



(a) Exact solution for Model 4.1.

(b) NAAM solution for Model 4.1

FIGURE 1. Presents Exact and NAAM solution graph of model 4.1 at $\wp = 1$, for $r = 0...1, \tau = 0...1$.

Model 4.2: we have considered multi-directional nonlinear dispersive shallow waters model (Fornberg–Whitham equation) as

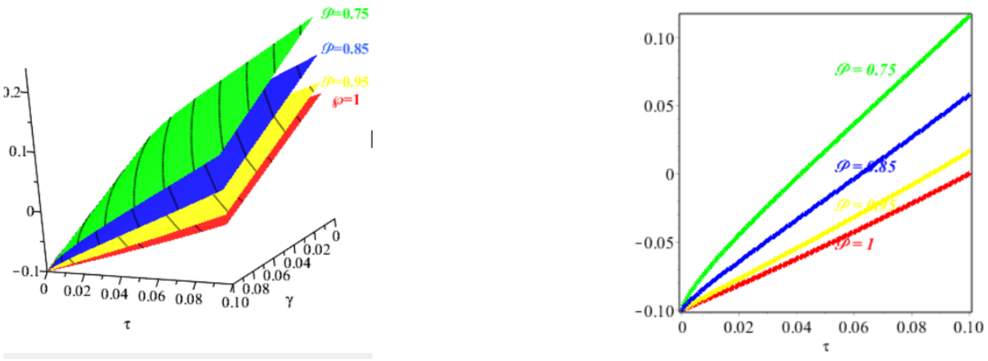
$$\zeta_\tau^{\wp}(r,\tau) - \zeta_{rr\tau}(r,\tau) + \pounds\zeta(r,\tau) = \pounds\zeta(r,\tau)\zeta_{rr\tau}((r,\tau) - \zeta(r,\tau)\zeta_(r) + \hbar\zeta_r(r,\tau)\zeta_{rr}, \quad \wp \in (0,1], \tag{40}$$

with initial source as

$$\zeta(r,0) = e^{\frac{1}{2}r},$$

at $\wp = 1$, the exact form solution is

$$\zeta(r,\tau) = e^{\frac{-1}{2}r+\frac{2\tau}{3}}.$$



(a) 3-D fractional order plots

(b) 2-D fractional order plot

FIGURE 2. Presents 3-D and 2-D fractional-order solution graph of model 4.1, for $r = 0...1, \tau = 0...1$.

To solve the equation 40, compare it with the equation (8), we get

$$\zeta_{\tau}^{\wp}(r,\tau) = \zeta_{rr\tau}(r,\tau) + \zeta(r,\tau) + \aleph(\zeta(r,\tau)), \quad \wp \in (0,1], \tag{41}$$

where, the non-linear term $\aleph(\zeta) = \pounds\zeta(r,\tau)\zeta_{rr\tau}((r,\tau) - \zeta(r,\tau)\zeta_{(}r) + \hbar\zeta_r(r,\tau)\zeta_{rr}(r,\tau)$.
The assumed approximate solution to the equation (2), is taken as

$$\zeta(r,\tau) = \sum_{\kappa=0}^{\infty} \zeta_{\kappa}(r,\tau). \tag{42}$$

The solution derived analytically from the equation (41), can be obtained by using the following procedure,

$$\zeta_{\tau}^{\wp}(r,\tau) = \lambda\left[\zeta_{rr\tau}(r,\tau) + \zeta(r,\tau) + \aleph(\zeta(r,\tau))\right], \quad \wp \in (0,1] \tag{43}$$

with initial source given by

$$\zeta(r,0) = e^{\frac{-1}{2}r}. \tag{44}$$

Assuming that equation (42), possesses a solution in the form of

$$\zeta_{\lambda}(r,\tau) = \sum_{\kappa=0}^{\infty} \lambda^{\kappa}\zeta_{\kappa}(r,\tau) \tag{45}$$

Taking Liouville-Caputo operator with respect to a special variable τ on both sides of equation (43), the property (5) and initial source (44), we get as

$$\zeta_{\lambda}(r,\tau) = \zeta(r,0) + \lambda I_{\tau}^{\wp}\left[\zeta_{rr\tau}(r,\tau) + \pounds\zeta(r,\tau) + \aleph(\zeta(r,\tau))\right], \tag{46}$$

with the help of definition (12), and equation (45), we have

$$\sum_{\kappa=0}^{\infty} \lambda^{\kappa}\zeta_{\kappa}(r,\tau) = \zeta(r,0) + \lambda I_{\tau}^{\wp}\left[\sum_{\kappa=0}^{\infty} \lambda^{\kappa}\left(\zeta_{\kappa rr\tau}(r,\tau) + \pounds\zeta_{\kappa}(r,\tau)\right) + \sum_{\kappa=0}^{\infty} \lambda^{\kappa}P_{\kappa}\right], \tag{47}$$

when equating terms of the same power of the parameter λ in equation (47), the recursive scheme transforms into of equation

$$\begin{cases} \zeta_0(r,\tau) = \zeta(r,0), \\ \zeta_1(r,\tau) = I_{\tau}^{\wp}\left[\zeta_{0rr\tau}(r,\tau) + \zeta_0(r,\tau) + P_0\right], \\ \zeta_{\kappa}(r,\tau) = I_{\tau}^{\wp}\left[\zeta_{(\kappa-1)rr\tau}(r,\tau) + \pounds\zeta_{(\kappa-1)}(r,\tau) + \sum_{\kappa=0}^{\infty} \lambda^{\kappa}P_{(\kappa-1)}\right]. \end{cases} \tag{48}$$

Thus evaluating for $\pounds = 1, \hbar = 3$,and simplifying, we obtain

$$\zeta_0(r,\tau) = e^{\frac{1}{2}r}, \tag{49}$$

$$\zeta_1(r,\tau) = e^{\frac{1}{2}r}\frac{\tau^{\wp}}{r(\wp+1)}, \tag{50}$$

$$\zeta_2(r,\tau) = \frac{-1}{8}e^{\frac{1}{2}r}\frac{\tau^{2\wp-1}}{r(2\wp)} + \frac{1}{4}e^{\frac{1}{2}r}\frac{\tau^{2\wp}}{r(2\wp+1)}, \tag{51}$$

$$\zeta_3(r,\tau) = \frac{-1}{32}e^{\frac{1}{2}r}\frac{\tau^{3\wp-2}}{r(3\wp-1)} + \frac{1}{8}e^{\frac{1}{2}r}\frac{\tau^{3\wp-1}}{r(3\wp)} + \frac{1}{8}e^{\frac{1}{2}r}\frac{\tau^{3\wp}}{r(3\wp+1)}, \tag{52}$$

$$\vdots$$

The NAAM solution is

$$\zeta(r,\tau) = \zeta_0(r,\tau) + \zeta_1(r,\tau) + \zeta_2(r,\tau) + \zeta_3(r,\tau) + \cdots \tag{53}$$

Substituting, equations (49,50,51,52) in equation (53), we get

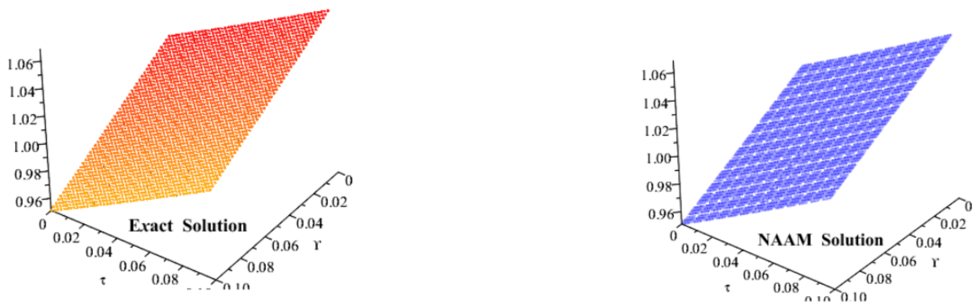
$$\zeta(r,\tau) = e^{\frac{1}{2}r} + e^{\frac{1}{2}r}\frac{\tau^{\wp}}{r(\wp+1)} + \frac{-1}{8}e^{\frac{1}{2}r}\frac{\tau^{2\wp-1}}{r(2\wp)} + \frac{1}{4}e^{\frac{1}{2}r}\frac{\tau^{2\wp}}{r(2\wp+1)} + \frac{-1}{32}e^{\frac{1}{2}r}\frac{\tau^{3\wp-2}}{r(3\wp-1)} + \frac{1}{8}e^{\frac{1}{2}r}\frac{\tau^{3\wp-1}}{r(3\wp)} + \frac{1}{8}e^{\frac{1}{2}r}\frac{\tau^{3\wp}}{r(3\wp+1)} + \ldots \tag{54}$$

Specifically for $\wp = 1$, we get as

$$\zeta(r,\tau) = e^{\frac{1}{2}r}\left(1 - \frac{5}{8}\tau + \frac{1}{8}\tau^2 + \cdots\right), \tag{55}$$

the obtained solution converge to exact solution form solution as

$$\zeta(r,\tau) = e^{(\frac{-1}{2}r + \frac{2\tau}{3})} \tag{56}$$



(a) Exact solution for Model 4.2.

(b) NAAM solution for Model 4.2

FIGURE 3. Presents Exact and NAAM solution graph of model 4.2 at $\wp = 1$**, for** $r = 0...1, \tau = 0...1$.

Model 4.3: We have consider the shallow water model(Whitham-Broer-Kaup model with time fractional order) as:

$$\begin{aligned} \zeta_\tau^{\wp}(r,\tau) + zeta_r(r,\tau)\zeta(r,\tau) + \xi_r(r,\tau) + b\zeta_{rr}(r,\tau) = 0, \\ \xi_\tau^{\wp}(r,\tau) + \zeta_r(r,\tau)\zeta(r,\tau) + a\xi_(rrr)(r,\tau) - b\xi_(rr)(r,\tau) = 0, \end{aligned} \quad \wp \in (0,1] \tag{57}$$

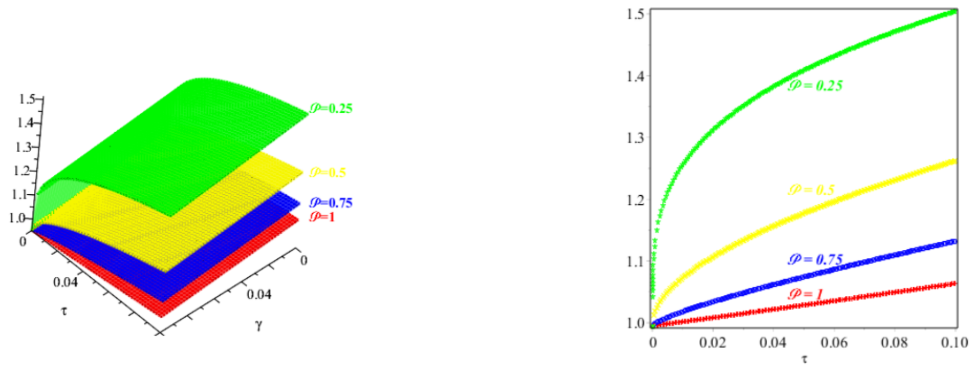
with the state value,

$$\zeta(r,0) = \hbar - 2\kappa(a+b^2)^{\frac{1}{2}}\coth(\kappa(r+r_0)),$$

$$\xi(r,0) = -2\kappa^2(a + b^2 + b(a+b^2)^{\frac{1}{2}})\cosh^2(\kappa(r+r_0)).$$

Taking into account the equation (57), Yields a solution represented by

$$\begin{aligned} \zeta_\lambda(r,\tau) = \sum_{\kappa=0}^{\infty}\lambda^\kappa\zeta_\kappa(r,\tau) \\ \xi_\lambda(r,\tau) = \sum_{\kappa=0}^{\infty}\lambda^\kappa\xi_\kappa(r,\tau) \end{aligned} \tag{58}$$



(a) 3-D fractional order plots

(b) 2-D fractional order plot

FIGURE 4. Presents 3-D and 2-D fractional-order solution graph of model 4.2, for $r = 0...1, \tau = 0...1$.

Employing the Liouville-Caputo operator with respect to a specified variable τ on both sides of the equation (57), the property (5), and initial sources, we get as

$$\begin{aligned} \zeta_\lambda(r,\tau) &= \zeta(r,0) + \lambda I_\tau^\wp \left[-\zeta(r,\tau)\zeta_(r)(r,\tau) - \xi_r(r,\tau) - b\zeta_{rr}(r,\tau)\right], \\ \xi_\lambda(r,\tau) &= \xi(r,0) + \lambda I_\tau^\wp \left[-\zeta(r,\tau)\zeta_(r)(r,\tau) - a\xi_(rrr)(r,\tau) + b\xi_(rr)(r,\tau)\right], \end{aligned} \tag{59}$$

with the help of (12), and equation (59), we obtain as

$$\begin{aligned} \sum_{\kappa=0}^{\infty} \lambda^\kappa \zeta_\kappa(r,\tau) &= \zeta(r,0) + \lambda I_\tau^\wp \left[-\sum_{\kappa=0}^{\infty} \lambda^\kappa P_\kappa - \xi_r(r,\tau) - b\zeta_{rr}(r,\tau)\right], \\ \sum_{\kappa=0}^{\infty} \lambda^\kappa \xi_\kappa(r,\tau) &= \xi(r,0) + \lambda I_\tau^\wp \left[-\sum_{\kappa=0}^{\infty} \lambda^\kappa P_\kappa^* - a\xi_{rrr}(r,\tau) + b\xi_{rr}(r,\tau)\right], \end{aligned} \tag{60}$$

equate, the terms with identical power of λ of equation (60), we get the approximated terms of the system in the form of recursive relation as

$$\begin{cases} \zeta_0(r,\tau) = \zeta(r,0), \\ \zeta_1(r,\tau) = \lambda I_\tau^\wp \left[-P_0 - \xi_r(r,\tau) - b\zeta_{rr}(r,\tau)\right], \\ \zeta_\kappa(\chi,r,\tau) = \lambda I_\tau^\wp \left[-P_{\kappa-1} - \xi_{(\kappa-1)r}(r,\tau) - b\zeta_{(\kappa-1)rr}(r,\tau)\right]. \end{cases} \tag{61}$$

$$\begin{cases} \xi_0(r,\tau) = \xi(r,0), \\ \xi_1(r,\tau) = \lambda I_\tau^\wp \left[-P_0^* - a\xi_{0rrr}(r,\tau) + b\xi_{0rr}(r,\tau)\right], \\ \zeta_\kappa(\chi,r,\tau) = \lambda I_\tau^\wp \left[-P_{\kappa-1}^* - a\xi_{(\kappa-1)rrr}(r,\tau) + b\xi_{(\kappa-1)rr}(r,\tau)\right]. \end{cases} \tag{62}$$

Thus, we obtain the terms of NAAM for equation (61), is

$$\zeta_0(r,\tau) = -2\kappa(a+b^2)^{\frac{1}{2}} \coth(\kappa(r+r_0)), \tag{63}$$

$$\zeta_1(r,\tau) = \frac{-2k^2\sqrt{(b^2+a)}\lambda}{cosh(k(r+r_0)^2-1)} \frac{t^\wp}{r(\wp+1)}, \tag{64}$$

$$\zeta_2(r,\tau) = \frac{-4k^3\lambda^2 cosh(k(r+r_0)) sinh(k(r+r_0))\sqrt{(b^2+a)}}{(cosh(k(r+r_0))^2-1)^2} \frac{t^{2\wp}}{r(2\wp+1)}, \tag{65}$$

$$\vdots$$

Thus, the other terms of NAAM for equation (62), is

$$\xi_0(r,0) = -2\kappa^2(a+b^2+b(a+b^2)^{\frac{1}{2}})\cosh^2(\kappa(r+r_0)), \tag{66}$$

$$\xi_1(r,\tau) = \frac{-4k^3\cosh(k(r+r_0))\sinh(k(r+r_0))\lambda(b^2+a+b\sqrt{(b^2+a)})}{(cosh(k(r+r_0))^2-1)^2} \frac{\tau^\wp}{r(\wp+1)}, \tag{67}$$

$$\xi_2(r,\tau) = \frac{-(4(2\cosh(k(r+r_0))^2\sqrt{(b^2+a)}b + 2\cosh(k(r+r_0))^2 b^2 + 2\cosh(k(r+r_0))^2 a + b\sqrt{(b^2+a)} + b^2 + a))\lambda^2 k^4}{(\cosh(k(r+r_0))^2-1)^2}, \frac{\tau^2\wp}{r(2\wp+1)} \tag{68}$$

$$\vdots$$

Thus, the NAAM solution become as

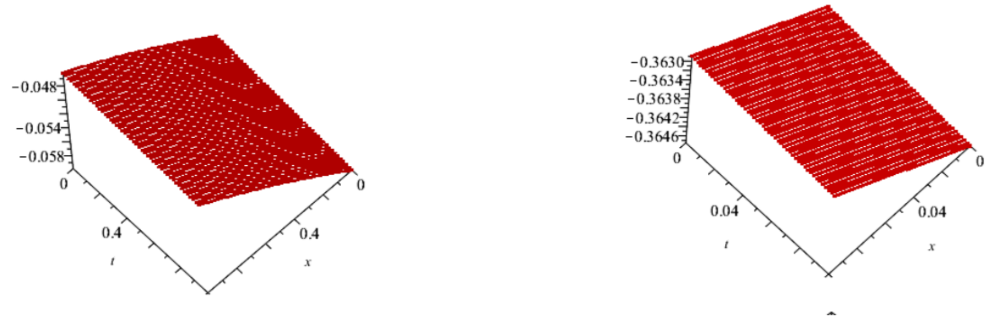
$$\begin{cases} \zeta(r,\tau) = \zeta_0(r,\tau) + \zeta_1(r,\tau) + \zeta_2(r,\tau) + \zeta_3(r,\tau) + \cdots \\ \xi(r,\tau) = \xi_0(r,\tau) + \xi_1(r,\tau) + \xi_2(r,\tau) + \xi_3(r,\tau) + \cdots \end{cases} \tag{69}$$

Substituting, equations (66,67,68) in equation 69, we get

$$\begin{cases} \zeta(r,\tau) = -2\kappa(a+b^2)^{\frac{1}{2}}\coth(\kappa(r+r_0)) + \frac{-2k^2\sqrt{(b^2+a)}\lambda}{cosh(k(r+r_0)^2-1)} \\ \frac{\tau^{\wp}}{r(\wp+1)} + \frac{-4k^3\lambda^2 cosh(k(r+r_0)) sinh(k(r+r_0))\sqrt{(b^2+a)}}{(cosh(k(r+r_0))^2-1)^2}\frac{\tau^{2\wp}}{r(2\wp+1)} \\ \xi(r,\tau) = -2\kappa^2(a+b^2+b(a+b^2)^{\frac{1}{2}})\cosh^2(\kappa(r+r_0)) + \frac{-4k^3\cosh(k(r+r_0))\sinh(k(r+r_0))\lambda(b^2+a+b\sqrt{(b^2+a)})}{(cosh(k(r+r_0))^2-1)^2}\frac{\tau^{\wp}}{r(\wp+1)} + \\ \frac{-(4(2\cosh(k(r+r_0))^2\sqrt{(b^2+a)}b+2\cosh(k(r+r_0))^2b^2+2\cosh(k(r+r_0))^2a+b\sqrt{(b^2+a)}+b^2+a))\lambda^2k^4}{(\cosh(k(r+r_0))^2-1)^2}\frac{\tau^2\wp}{r(2\wp+1)} + \cdots \end{cases} \tag{70}$$

For $\wp = 1$, we get as

$$\begin{cases} \zeta(r,\tau) = -2\kappa(a+b^2)^{\frac{1}{2}}\coth(\kappa(r+r_0)) + \frac{-2k^2\sqrt{(b^2+a)}\lambda}{cosh(k(r+r_0)^2-1)} \\ \tau + \frac{-4k^3\lambda^2 cosh(k(r+r_0)) sinh(k(r+r_0))\sqrt{(b^2+a)}}{(cosh(k(r+r_0))^2-1)^2}\frac{t^2}{r(3)} \\ \xi(r,\tau) = -2\kappa^2(a+b^2+b(a+b^2)^{\frac{1}{2}})\cosh^2(\kappa(r+r_0)) + \frac{-4k^3\cosh(k(r+r_0))\sinh(k(r+r_0))\lambda(b^2+a+b\sqrt{(b^2+a)})}{(cosh(k(r+r_0))^2-1)^2}t + \\ \frac{-(4(2\cosh(k(r+r_0))^2\sqrt{(b^2+a)}b+2\cosh(k(r+r_0))^2b^2+2\cosh(k(r+r_0))^2a+b\sqrt{(b^2+a)}+b^2+a))\lambda^2k^4}{(\cosh(k(r+r_0))^2-1)^2}\frac{\tau^2}{r(3)} + \cdots \end{cases} \tag{71}$$



a) NAAM solution of $\xi(r,\tau)$ for Model4.3. (b) NAAM solution of $\xi(r,\tau)$ for Model 4.3

FIGURE 5. **Presents NAAM solutions ($\zeta(r,\tau)$, $\xi(r,\tau)$) graph of model 4.1 at $\wp = 1$, for** $r = 0...1, \tau = 0...1.$

5. RESULTS AND DISCUSSION

This section discusses the results obtained using the new approximate analytical method (NAAM) for shallow water models with fractional orders (23), (40), and (57). The analytical solutions derived through NAAM are presented for these models with fractional orders. When the fractional order becomes an integer, the series solution converges to the exact solution. The comparison between the approximated and exact solutions is depicted in Figures 1 and 2, demonstrating the validity and feasibility of NAAM. This illustration reflects the convergence behavior of the proposed method. Additionally, the fractional order solutions for Models (23) and (40) are explored. The 3D graph in Figure 3 and the 2D graph in Figure 4 showcase different results for various fractional orders. These multi-order solutions capture diverse flow phenomena, with fractional orders aligning better with real physical phenomena, providing an objective solution to the models.

6. CONCLUSION

This paper conducts an analytical investigation of fractional fluid flow models (shallow water wave models) with Caputo fractional derivatives. A new approximate analytical technique is applied to derive results for several illustrative models. Comparisons are made between the results obtained for both fractional and integer orders of the problems. Solution graphs are plotted for these different orders, showing close agreement between the exact and NAAM results.

Fractional-order solutions offer valuable insights into the dynamics of the investigated problems. The simplicity of the suggested method highlights its potential for solving other fractional-order problems in natural phenomena due to its straightforward procedure. Finally, it is concluded that NAAM provides better accuracy and reduces computational work. For application purposes, NAAM can be extended to other physical phenomena arising in the field of applied sciences and engineering.

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CONFLICT OF INTEREST

Authors declare no conflict of interest

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