Essentially Retractable Modules Relative To A Submodule And Some Generalizations

Shukur Neamah Al-Aeashi^{1,*}, Fatimah Hussein Al-Bakaa²

¹Department Of Mathematics, College of Education, University Of Kufa, Najaf, Iraq

²Department Of Mathematics, College Of Computer Science And Mathematics, University Of Kufa, Najaf, Iraq

*Corresponding Author: Shukur Neamah Al-Aeashi

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ABSTRACT: R is a ring with unity, and all modules are unitary right R -modules. The concept of compressible modules was introduced in 1981 by Zelmanowitz, where module M is called compressible if it can be embedded in any nonzero submodule A of M . In other words, M is a compressible module if for each nonzero submodule A of M , $f \in \text{Hom}(M, A)$ exists, such that f is monomorphism. Retractable modules were introduced in 1979 Khuri, where module M is retractable if $\text{Hom}(M, A) \neq 0$ for every nonzero submodule A of M . We define a new notion, namely, essentially retractable module relative to a submodule. In addition, new generalizations of compressible modules relative to a submodule are introduced, where module M is called compressible module relative to a submodule N of M . If for all nonzero submodule K of M contains N , then a monomorphism $f \in \text{Hom}(M, K)$ exists. Some basic properties are studied and many relationships between these classes and other related concepts are presented and studied. We also introduce another generalization of retractable module, which is called small kernel retractable module.

Keywords: Compressible module relative to a submodule, retractable module relative to a submodule, essentially retractable module relative to a submodule, small kernel retractable module.

1. INTRODUCTION

R is a ring with unity, and all modules are unitary right R -modules. The concept of compressible modules was introduced in 1981 by Zelmanowitz, where a module M is called compressible if it can be embedded in any nonzero submodule A of M . In other words, M is a compressible module if for each nonzero submodule A of M , $f \in \text{Hom}(M, A)$ exists, such that f is a monomorphism. Retractable modules were introduced in 1979 by Khuri, where module M is retractable if for every nonzero submodule A of M , $\text{Hom}(M, A) \neq 0$ exists [16]. Evidently, the converse of this concept may not be true. For example, the Z -module Z_4 is retractable, but it is incompressible. Moreover retractable modules, compressible modules, and some generalizations have been studied by several authors, namely, [5, 7, 9, 11, 17]; a dual retractable concept in 2006, a coretractable module in [8]. However, Amini [6] studied this class of modules, where M is called coretractable if for all appropriate submodule A of M , $0 \neq f \in \text{Hom}(M/A, M)$ exists. Then, more authors studied this concept and its generalizations, namely, [4, 12–14].

Al-Aeashi and Al-Bakaa introduced the concept of retractable module relative to a submodule, where for every appropriate submodule K of M containing N , a nonzero homomorphism $f : M \rightarrow K$ exists, as shown in [2]. Then, [3] introduced the concept of compressible module relative to a submodule, where for every appropriate submodule K of M containing N , a monomorphism $f : M \rightarrow K$ exists. In this work, we introduce a new notion of retractable module relative to a submodule, where an R -module M is called an essentially retractable module relative to a submodule N of M (essentially N -retractable). If for all submodule K of M , where $N \leq K$, a nonzero homomorphism $f : M \rightarrow K$ exists,

such that $f(M) \leq_e M$. Some basic properties of these notions are provided. In addition, we introduce the concept of small kernel retractable module, where an R -module M is called small kernel retractable module. In summary, in sk -retractable module, if for each nonzero submodule N of M , $f: M \rightarrow N$ exists, such that $f \ll M$. We shall denote $\leq, \ll, \leq_e$, and $End(M)$ for the submodule, small submodule, essential submodule, and the endomorphism of a module M , respectively.

2. BASIC CONCEPTS:

Definition 2.1. [15] A submodule A of a module M is called small (= superfluous) in M , respectively large (= essential) in M , notational $(A \ll M) A \leq_e M \Leftrightarrow \forall U \leq M [A + U = M \Rightarrow U = M]$ respectively $\forall U \leq M [U \cap A = 0 \Rightarrow U = 0]$

An R -module M is called uniform, if $M \neq 0$ and every nonzero submodule of M is essential in M [15].

Definition 2.2. An R -module M is called retractable if for all non-zero submodule N of M , there exists a non-zero homomorphism $f: M \rightarrow N$ [16].

Definition 2.3. An R -module M is called compressible if for all non-zero submodule N of M , there exists a non-zero monomorphism $f: M \rightarrow N$ [16].

It is clear every compressible module is retractable. There are some generalizations of these concepts as follows:

Definition 2.4. An R -module M is called small compressible if M can be embedded in each of its non-zero small submodule [17].

Equivalently, M is small compressible if there exists a monomorphism from M into N whenever $0 \neq N \ll M$. Where a submodule K of M is called small in M ($K \ll M$), in case $M = K + L$ implies that $L = M$ for any submodule L of M . A ring R is called small compressible if the R -module R is small compressible. That is R can be embedded in any of its non-zero small ideal.

Definition 2.5. An R -module M is called Completely slightly compressible if for all proper submodule N of M , there exists $f: M \rightarrow N$, $\ker f \cap N = 0$.

Definition 2.6. A module M is called small retractable if $\text{Hom}(M, N) \neq 0$ for each non-zero small submodule N of M , [17].

Definition 2.7. A module M is called prime if $\text{ann}(x) = \text{ann}(y)$ for each non-zero element x and y in M .

3. THE MAIN RESULTS

In this section, we introduce a new class of module namely essentially retractable module relative to a submodule and studied some properties on it.

Definition 3.1. Let M be an R -module and N be a submodule of M . M is called essentially N -retractable module if for all $K \geq N$, there exists homomorphism $f: M \rightarrow K$, such that $f(M) \leq_e M$.

Example 3.2. Let M be an R -module and $N \leq M$ Then if every essentially N -retractable module M is N -retractable module.

Proof. Let $K \geq N$, so there exists a non-zero homomorphism $f: M \rightarrow K$ such that $f(M) \leq_e M$ (since M is an essentially N -retractable module) and hence M is an N -retractable module.

Remark 3.3. The converse is not true in general, for example Consider the Z_{12} as Z -module. It is clear Z_{12} is $(\bar{0})$ -retractable module but not essentially N -retractable module. Since $(\bar{6})$ contain $(\bar{0})$ but there is no non-zero homomorphism f from Z_{12} into $(\bar{6})$ such that $\text{Im} f$ is essential submodule of Z_{12} .

Proposition 3.4. Let M be an R -module, N be a submodule of M , if M is uniform module, then M is essentially N -retractable module if and only if M is N -retractable module.

Proof. $(\Rightarrow)$ By obvious

$(\Leftarrow)$ Let $K \geq N$, by hypothesis $\exists f: M \rightarrow K$, $f \neq 0$ $\text{Im} f \leq K$ and hence $\text{Im} f \leq M$ thus $\text{Im} f \leq_e M$ since M is uniform. Thus M is essentially N -retractable module.

Proposition 3.5. Let M is an essentially N_1 -retractable module and M is an essentially N_2 -retractable module. Then M is an essentially $N_1 + N_2$ -retractable module.

Proof. Let $N_1 + N_2 \leq K$, so $N_1 \leq K$ and $N_2 \leq K$ but M is N_1 -retractable module so there exists $f : M \rightarrow K$ and $f(M) \leq_e M$, likewise M is N_2 -retractable module, so M is an essentially $N_1 + N_2$ -retractable module.

Proposition 3.6. Let M be an R -module and N, F be two submodules of M . If M is an essentially N -retractable module and $F \geq N$, then M is an essentially F -retractable module.

Proof. Let $K \geq F$. Hence $K \geq N$, but M is an essentially N -retractable module so there exists $0 \neq f : M \rightarrow K$ such that $f(M) \leq_e M$. Therefore M is an essentially F -retractable module.

Proposition 3.7. Let M be a module and $(N_i)_{i \in I}$ be a family of submodule of M , if M is essentially N_i -retractable module for each $i \in I$. Then M is essentially $\sum_{i \in I} N_i$ -retractable module.

Proof. Let K be a non-zero submodule such that $\sum_{i \in I} N_i \leq K$ hence $N_i \leq K$ for each $i \in I$. Since M is essentially N_i -retractable module, so there exists non-zero homomorphism $f : M \rightarrow K$ such that $\text{Im} f$ is essential submodule of M , therefore M is essentially $\sum_{i \in I} N_i$ -retractable module.

Proposition 3.8. Let $M = M_1 \oplus M_2$, such that M_1 is essentially N_1 -retractable module and M_2 is essentially N_2 -retractable module. Then M is essentially $N_1 \oplus N_2$ -retractable module provided M is duo module.

Proof. Let $K \leq M$ such that $K \geq N_1 \oplus N_2$, since M is duo module hence $K = K_1 \oplus K_2$ where $K_1 = K \cap M_1$, $K_2 = K \cap M_2$ since $K_1 \oplus K_2 \geq N_1 \oplus N_2$, then $K_1 \geq N_1$ and $K_2 \geq N_2$. Hence $\exists f : M_1 \rightarrow K_1$ and $g : M_2 \rightarrow K_2$ and both f, g are non-zero homomorphisms with $f(M_1) \leq_e M_1$ and $g(M_2) \leq_e M_2$.

Define $h : M_1 \oplus M_2 \rightarrow K_1 \oplus K_2$ by $h(m_1, m_2) = (f(m_1), g(m_2))$, then $h(M_1 \oplus M_2) = f(M_1) \oplus g(M_2) \leq_e M_1 \oplus M_2$. By [2, Proposition 17.4] Thus $M_1 \oplus M_2$ is essentially $N_1 \oplus N_2$ -retractable module.

Proposition 3.9. Let $M = M_1 \oplus M_2$, and M_1, M_2 be two R -module such that $\text{ann} M_1 + \text{ann} M_2 = R$. If M_1 is essentially N_1 -retractable module and M_2 is essentially N_2 -retractable module, then M is essentially $N_1 \oplus N_2$ -retractable module.

Proof. Let $K \geq N_1 \oplus N_2$. Since $R = \text{ann} M_1 + \text{ann} M_2$, then $K = K_1 \oplus K_2$ for some $K_1 \leq M_1$, $K_2 \leq M_2$ where $K_1 = K \cap M_1$ and $K_2 = K \cap M_2$ since $K_1 \oplus K_2 \geq N_1 \oplus N_2$, then $K_1 \geq N_1$ and $K_2 \geq N_2$. Hence $\exists f : M_1 \rightarrow K_1$ and $g : M_2 \rightarrow K_2$ where $f, g \neq 0$ and $f(M_1) \leq_e M_1$ and $g(M_2) \leq_e M_2$.

Define $h : M_1 \oplus M_2 \rightarrow K_1 \oplus K_2$ by $h(m_1, m_2) = (f(m_1), g(m_2))$, then $h(M_1 \oplus M_2) = f(M_1) \oplus g(M_2) \leq_e M_1 \oplus M_2$. Thus $M_1 \oplus M_2$ is essentially $N_1 \oplus N_2$ -retractable module.

Now, we shall give another new generalization of retractable module relative to a submodule as follows:

Definition 3.10. An R -module M is called small kernel retractable module (briefly, sk-retractable module) if for each non-zero submodule N of M , there exists $f : M \rightarrow N$ such that $\ker f \ll M$.

Remarks and Examples 3.11. 1) Each a compressible module is an sk-retractable module

Proof. Let M be a compressible module, and $N \neq 0$ be a submodule of M , by compressibility condition there exists $f : M \rightarrow N$ monomorphism thus $\ker f = 0$ and that mean $\ker f \ll M$. Thus M is an sk-retractable module.

The converse of part(1) is not true in general, for example, consider the Z -module Z_4 is sk-retractable module but not compressible module.

2) Every small kernel retractable module is retractable module

Proof. Let M be a small kernel retractable module, hence for each non-zero submodule N of M there exists non-zero homomorphism $f : M \rightarrow N$ such that $\ker f \ll M$, and hence M is retractable module.

The converse of this is not true in general, for example, see Z_6 as Z -module is clear that retractable module but it is not sk-retractable module.

3) Let M be an R -module. If M is simple then M is sk-retractable module if and only if M is retractable module.

Proof. ($\implies$) It is clear by part(2)

($\impliedby$) Let $0 \neq N \leq M$, by retractability there exists $f : M \rightarrow N$, $f \neq 0$, clear that $\ker f \leq M$ but M is simple module hence either $\ker f = 0 \ll M$ or $\ker f = M$, so $f = 0$ and that contradiction. Then M is sk-retractable module.

The condition (simple module) in part(3) is necessary for example, consider the converse in part(2) clear that Z_6 as Z -module is retractable module but not sk-retractable module, since Z_6 is not simple that is mean Z_6 has no small submodule.

4) One easy can see there is no relation between sk-retractable module and semisimple consider Z_4 as Z -module is sk-retractable module but Z_4 is not semisimple while Z_6 as Z -module is semisimple but not sk-retractable module as the converse in part(2).

Proposition 3.12. Every direct summand of sk-retractable module is also sk-retractable module.

Proof. Let $N \leq K \leq M$ where $K \leq^{\oplus} M$. Since M is sk-retractable, there exists $f : M \rightarrow N$ such that $\ker f \ll M$ but we have the inclusion map from K into M , $i : K \rightarrow M$ so, $foi : K \rightarrow N$ is non-zero homomorphism and $\ker(foi) \subseteq \ker f$ and $\ker f \ll M$, then $\ker f(foi) \ll M$. By proposition(3.4) on the other hand $\ker f(foi) \subseteq K$ and $K \leq^{\oplus} M$, so $\ker(foi) \ll K$ see [19, Proposition 2.2(6)]. Thus K is sk-retractable module.

Under another condition we can obtain the equivalence between sk-retractable module and retractable module as the following proposition:

Proposition 3.13. Let M be an R -module. If M is hollow module, then M is sk-retractable module if and only if M is retractable module.

Proof. ($\implies$) Clear

($\impliedby$) Let N be a non-zero submodule of M , by retractability condition we always have a non-zero homomorphism $f : M \rightarrow N$ where $\ker f \leq M$, but M is hollow module by hypothesis of hollow module therefore $\ker f \ll M$ and hence M is sk-retractable module.

The condition hollow module in proposition(3.13) is necessary for example Z_6 is retractable module as Z -module but Z_6 is not hollow module, therefore is not sk-retractable module .

Proposition 3.14. Let R be a commutative ring, then a finitely generated non-zero R -module is compressible if and only if M is uniform prime module.[10, lemma26.2.9]

Proposition 3.15. Let R be a commutative ring, and M be an R -module, if M is uniform prime then M is sk-retractable module.

Proof. Let M be a uniform and prime, then by proposition(3.13) M is compressible module and by using, part(1)) we obtain M is sk-retractable module.

Recall that an R -module M is called distributive if the lattice of its submodules is distributive, that is $L \cap (N + K) = (L \cap N) + (L \cap K)$.

Equivalently, $L + (N \cap K) = (L + N) \cap (L + K)$ for all submodules L, N , and K of M " [1]

Proposition 3.16. Let M be a distributive module and $M = M_1 \oplus M_2$. If M_1 and M_2 are sk-retractable module, then M is sk-retractable module, provided $N \cap M_1 \neq 0$ and $N \cap M_2 \neq 0$ for all $0 \neq N \leq M$.

Proof. Let $0 \neq N \leq M$. As M is distributive so, $N = (N \cap M_1) \oplus (N \cap M_2)$. Let $(N \cap M_1) = N_1$ and $(N \cap M_2) = N_2$. Since M_1 and M_2 are sk-retractable modules, there exist $f : M_1 \rightarrow N_1$ and $g : M_2 \rightarrow N_2$ such that $f \neq 0$, $g \neq 0$ and $\ker f \ll M_1$ and $\ker g \ll M_2$. Define $h : M \rightarrow N$, where $N = N_1 \oplus N_2$ by $h(x,y)=(f(x),g(y))$. It is clear is well-defined homomorphism. It is easy to see that $\ker h = \ker f \oplus \ker g$ by [15, lemma4.3.2(2)], hence $\ker h \ll M_1 \oplus M_2$. Thus M is sk-retractable module.

By using this proposition we get some corollaries as follows:

Corollary 3.17. Let R be a commutative ring. Let M be a uniform R -module, if every non-zero 2-generated submodule of M is prime, then M is sk-retractable module.

Proof. By hypothesis we obtain M is prime from [10, lemma.26(2.5)] and hence by Proposition(3.16) M is compressible, therefore by proposition(3.15) M is sk-retractable module.

Corollary 3.18. Let R be a commutative ring and M be a uniform R -module if $\frac{M}{K}$ is prime of each proper complement K of M , then M is sk-retractable module.

Proof. By using [10, lemma.26(2.5)], M is prime and hence by Proposition(3.16), M is sk-retractable module.

REFERENCES

- [1] M. S. Abbas, On fully stable Modules, Ph. D. Thesis, University of Baghdad, 1990.
- [2] S. N. Al-Aeashi and F. H. Al-Bakaa, "Retractable modules relative to a submodule and some generalizations", *IOP Conf. Ser.: Mater. Sci. Eng.* 928 042014, 2020, DOI: 10.1088/1757-899X/928/4/042014
- [3] S. N. Al-Aeashi and F. H. Al-Bakaa, "Compressible modules relative to a submodule and some generalizations", *Test Engineering and management*, Vol. 83(2020), 30117 -30122, 2020. DOI: 10.24996/IJS.2019.60.6.21
- [4] S. N. Al-Aeashi, F. D. Shyaa, I.M.A. Hadi. "Essentially and weakly essentially coretractable modules". *J. Phys.: Conf. Ser.*; 1234, 012107,2019. Doi:10.1088/1742-6596/1234/1/012107.
- [5] A. M. A. Al-Hosainy and H. Kadhim. "A Co-compressible Module". *International Research Journal of Scientific Findings* 1(6), 2014.
- [6] B. Amini, M. Ershad, H. Sharif. "Coretractable modules". *J. Aust. Math. Soc.*; 86(3): 289304, 2009.
- [7] M. Baziar, "Semi-essentially compressible modules and strongly semiprime rings", *International Journal of Algebra* 3 (14) 685692, 2009.
- [8] J. Clark, C. Lomp, N. Vanaja, R. Wisbauer. *Lifting modules. Supplements and Projectivity in Module Theory*, Frontiers in Mathematics, Birkhäuser, Basel, Boston, Berlin, 2006.
- [9] S. Ecevit and M. T. Kosan, "On Rings All Of Whose Modules are Retractable", *Arch. Math. (Brno)*, 45(1), 71-74, 2009.
- [10] R. Elman, *Lectures on Abstract Algebra*, University of California, Los Angeles, U.S.A. 1999.
- [11] I.M. Ali Hadi and S. N. Al-aeashi. "Y-coretractable and Strongly Y-Coretractable Modules". *Asian Journal of Applied Sciences*, Vol. 5, No. 2, Pp.427-433, 2017.
- [12] I. M. Hadi, S. N. Al-aeashi and F. D. Shyaa. t-Essentially and weakly t-essentially coretractable modules, 2019.
- [13] I. M. Hadi and S. N. Al-aeashi. "Weakly coretractable modules". *J. Phys.: Conf. Ser.* 1003 012061, 2018.
- [14] I. M. Hadi and S. N. Al-aeashi. "Strongly Coretractable modules". *Iraqi journal of sciences* 58 2C pp 1069-1075, 2017.
- [15] I.M.Ali Hadi and Sh. N. Al-aeashi. "Strongly Coretractable Modules and Some Related Concepts". *Journal of advances in Mathematics*, Vol.12, No.12, 6881-6888, 2016.
- [16] I. M. Hadi and S. N. Al-aeashi. "Studying Some Results about Completely Coretractable Ring (CC-ring)". *Global Journal of Mathematics* Vol. 10, No.1, Pp. 644-647, 2017.
- [17] S. M. Layla and H. M. Israa. "Small Compressible Modules And Small Retractable Modules". *International Journal Of Applied Mathematics & Statistical Sciences*, 5(2) 15-28, 2016.
- [18] A. R. Moniri, Y. Talebi, "Coretractable modules relative to a submodule", *J. Algebra Comb. Discrete Appl.* 6(2): 95103, 2019.
- [19] Wisbauer, Robert. *Foundations of module and ring theory*. Routledge, 2018.