Chaotic Dynamics in the 2D System of Nonsmooth Ordinary Differential Equations

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ABSTRACT: Over the last decade, the chaotic behaviors of dynamical systems have been extensively explored. Recently, discovering or developing a 2D system of ordinary differential equations (ODEs) capable of exhibiting chaotic dynamical behaviors is an attractive research topic. In this study, a chaotic system with a 2D system of nonsmooth ODEs has been developed. This system can exhibit chaotic dynamical behaviors. Its main dynamical behaviors, including time-series trajectories, phase portraits of attractors, and equilibria and their stability, have been investigated. The developed system has been verified by an excessive variety of fascinating chaotic behaviors, such as chaotic attractor, symmetry, sensitivity to initial conditions (ICs), fractal dimension, autocorrelation, power spectrum, Lyapunov exponent, and bifurcation diagram. Analytical and numerical simulations are used to study the dynamical behaviors of such a system. The developed system has extreme sensitivity to ICs, a fractal dimension of more than 1.8 and less than 2.05, an autocorrelation fluctuating randomly about an average of zero, a broadband power spectrum, and one positive Lyapunov exponent. The obtained numerical simulation results have proven the capability of the developed 2D system for exciting chaotic dynamical behaviors.

Keywords: Chaos; Nonlinear dynamics; 2D system; Nonautonomous system; Nonsmooth ODE system

1. INTRODUCTION

In a deterministic system, chaos is an aperiodic long-term behavior that is extremely dependent on initial conditions (ICs) and a small change in system parameters [1]. Although chaos has a noise-like appearance, it is deterministic in nature, whereas noise is random [2]. Chaos is a term used to describe nonlinear systems that exhibit chaotic behaviors [3]. Given that chaotic systems follow the butterfly effect, which states that even a small flapping of a butterfly's wings can generate a hurricane, they are extremely sensitive to ICs [4].

Research on chaotic systems has had a significant attraction since Lorenz recognized chaos theory in 1963 [5]. Over the last few decades, chaos phenomena in nonlinear systems have been widely exploited in engineering, science, and applied mathematics [6]. Chaotic complex systems are critical in a wide range of scientific and engineering applications, such as encryption, induction motors for drilling, chemical reactor systems, bridge wing design, aircraft control systems, weather prediction systems, PLL systems, memristive circuits, secure communication schemes, and cryptography systems [7].

Chaotic systems can be represented by differential equations in which any two solutions for two relatively similar conditions are extremely different at the same moment. Many chaotic systems are designed to generate chaos for certain goals, such as Rössler, Lorenz, Lü, and Chen systems [8]. Attractors are nothing more than a set of locations in phase space that are invariant under the dynamics generated by these chaotic systems. These attractors are nonperiodic in nature [9].

Chaos is impossible in a 2D system of autonomous smooth ODEs. That is, chaotic dynamical behaviors can only be found in 3D (or more) systems of autonomous smooth ODEs, such as the Lorenz system. Meanwhile, 2D systems of nonautonomous systems, nonsmooth ODEs, and forced systems can excite chaos, such as a forced pendulum. This fact has been highlighted in many main references in the chaotic subject area, such as [10–14]. However, systems with a 2D system of nonsmooth ODEs, which can excite chaos, are still limited.

Given that the Poincaré–Bendixson theorem excludes chaos exhibition in 2D continuous dynamical systems, this claim piques people’s interest. Dixon et al. (1993) developed a 2D system and showed that their system is a counterexample because this mentioned theorem’s assumptions are violated [15]. The Dixon system is frequently referred to as an example of a 2D system that is capable of exhibiting chaos phenomenon. Alvarez-Ramirez et al. (2005) demonstrated that the seemingly chaotic behavior is simply due to the existence of two elliptic sectors with an infinite number of homoclinic orbits. Thus, a chaotic dynamical behavior can be exhibited by a 2D system [16]. Sprott (2010) investigated the chaotic dynamical behaviors of 2D systems. He also showed that such systems can excite chaos while containing nonsmooth terms (e.g., logarithm terms), which are responsible for exhibiting chaotic behaviors [17].

In this work, a 2D nonautonomous (forced) system has been developed to be a nonsmooth ODE system that can display chaotic dynamical behaviors. Several common tools of the chaotic system analysis are used to investigate nonlinear dynamical behaviors, analytically and numerically. These behaviors are analyzed to validate the chaotic behaviors of the developed system.

2. 2D NONSMOOTH CHAOTIC SYSTEM

In 1993, Dixon et. al. introduced a 2D ODE continuous-time system, which depicts the dynamical behavior of a neutron star’s magnetic field in a model, as in the following Equation (1) [18].

$$\begin{aligned}\dot{x} &= \frac{xy}{x^2 + y^2} - \alpha x \\ \dot{y} &= \frac{y^2}{x^2 + y^2} - \beta y + \beta - 1\end{aligned}\quad (1)$$

where α and β represent the system parameters.

Despite the seeming violation of the Poincaré–Bendixson theorem, System (1) preserves the chaotic behaviors.

2.1 POINCARÉ–BENDIXSON THEOREM:

The attractor for any smooth 2D-bounded continuous-time autonomous system is either a stable equilibrium or a limit cycle [19].

However, System (1) cannot be considered an autonomous system because it contains a direct current (D.C.) value(s) (β and/or 1), where the system includes any explicit time dependence (e.g., $\sin(t)$) or/and D.C. source made it a nonautonomous (forced) system [20].

3. DEVELOPED SYSTEM

In a 2D system of autonomous ODEs, chaos is impossible. That is, chaotic dynamical behaviors exist only in a 3D system of autonomous ODEs or in a 2D system of nonautonomous ODEs (forced) systems or nonsmooth systems. This fact has been stated in many main references in the chaotic subject area, as mentioned in the Introduction section. In our novel contribution, we developed a 2D system of nonautonomous ODEs to be nonsmooth without a D.C. value (constant value) source, which can exhibit chaotic dynamical behaviors. In the chaotic subject area currently, limited research has introduced 2D systems of nonsmooth ODEs; for example, [21]. Based on the system described by Equation (1), we developed it to be nonsmooth without D.C. source, as described in the following Equation (2).

$$\begin{aligned}\dot{x} &= \frac{xy}{x^2 + y^2} \\ \dot{y} &= \frac{y^2}{x^2 + y^2} - ay - b|x|\end{aligned}\quad (2)$$

where x and y are the state variables, a and b are the system positive parameters. System (2) behaves chaotically over a wide range of system parameters a and b . In the numerical simulation, the parameters were chosen to be $a = b = 10$, with ICs as $(x_0, y_0) = (0.01, 0.01)$. Figs. 1 and 2 show the time series of system states and phase portraits as projections on a 2D plane, respectively.

Figs. 1 and 2 clearly seem to illustrate a violation of the Poincaré–Bendixson theorem. Evidently, the attractor of System (2) is not confined to a fixed point or a periodic orbit but its complex dynamical behavior.

In the dynamical performance of nonlinear systems, equilibrium points play a fundamental role. To analyze nonlinear systems, locating these points and investigating their types are necessary. State Equation (2) is equaled to zero, as in Equation (3), to determine the equilibrium points (equilibria) of the developed system.

$$\begin{aligned}\dot{x} &= \frac{xy}{x^2 + y^2} = 0 \\ \dot{y} &= \frac{y^2}{x^2 + y^2} - ay - b|x| = 0\end{aligned}\quad (3)$$

FIGURE 1. State variables of System (2) with respect to time

FIGURE 2. Chaotic attractor phase portraits of System (2)

Its clear solution of Equation (3) can be achieved only at $x = 0$ and $y = 0$. Thus, System (2) only has a single equilibrium point at the origin $(x^*, y^*) = (0, 0)$. To determine the stability of the equilibrium point, the Jacobian matrix of System (2) is evaluated, as described by Equation (4), where the linearizing approach is used.

$$J = \begin{bmatrix} \frac{y(x^2 + y^2) - 2yx^2}{(x^2 + y^2)^2} & \frac{x(x^2 + y^2) - 2xy^2}{(x^2 + y^2)^2} \\ \frac{-2xy^2}{(x^2 + y^2)^2} - b\text{sign}(x) & \frac{2y(x^2 + y^2) - 2y^3}{(x^2 + y^2)^2} - a \end{bmatrix}\quad (4)$$

Then, the following equation is used to compute the eigenvalues corresponding to the equilibrium points [22].

$$(\lambda I - J(x^*, y^*)) = 0 \quad (5)$$

In Equation (4), directly substituting equilibrium point $(x^*, y^*) = (0, 0)$ is impossible. Therefore, we used the L'Hospital's rule as in the following. Consider that the Jacobian matrix elements are symbolized by J_{11} , J_{12} , J_{21} , and J_{22} . Then, we can calculate these elements at the equilibrium point as shown below.

$$\begin{aligned} J_{11}(0,0) &= \lim_{x,y \rightarrow 0,0} \frac{y(x^2 + y^2) - 2yx^2}{(x^2 + y^2)^2} \\ J_{12}(0,0) &= \lim_{x,y \rightarrow 0,0} \frac{x(x^2 + y^2) - 2xy^2}{(x^2 + y^2)^2} \\ J_{21}(0,0) &= \lim_{x,y \rightarrow 0,0} \frac{-2xy^2}{(x^2 + y^2)^2} - \text{bsign}(x) \\ J_{22}(0,0) &= \lim_{x,y \rightarrow 0,0} \frac{2y(x^2 + y^2) - 2y^3}{(x^2 + y^2)^2} - a \end{aligned} \quad (6)$$

Given that two variables are involved, a polar coordinate system is needed to apply the L'Hospital's rule (multiple variable limits), where $x = r\cos(\theta)$ and $y = r\sin(\theta)$, and $(x, y) \rightarrow (0, 0)$ gives limits $r \rightarrow 0$ and limits for θ . Thus, the results of all the Jacobian elements are obtained as follows:

$$\begin{aligned} J_{11}(0,0) &= 0 \\ J_{12}(0,0) &= 0 \\ J_{21}(0,0) &= \begin{cases} 1 & ; x \rightarrow 0^+ \\ -1 & ; x \rightarrow 0^- \end{cases} \\ J_{22}(0,0) &= -a \end{aligned} \quad (7)$$

On the basis of Equation (5), the eigenvalue corresponding to the equilibrium point of System (2) is calculated as $\lambda_1 = -a$ and $\lambda_2 = 0$. Therefore, the equilibrium point can be classified as critically (marginally) stable, which cannot be considered asymptotically stable. That is, this equilibrium point is unstable, as practically all trajectories in the neighborhood do not converge to it [23, 24]. As a result, this fixed point is responsible for exciting self-excited chaotic attractors [25].

4. CHAOTIC DYNAMICAL BEHAVIOR PROOF

Some tools are investigated and discussed to show the chaotic dynamical behaviors of System (2).

4.1 SYMMETRY

The system evidently remains invariant during the transformation:

$$(x, y) \rightarrow (-x, y) \quad (8)$$

That means, if x and y are the solutions of System (2), then $-x$ and y are also the solutions for it.

4.2 SENSITIVITY TO ICS

The chaotic systems are extremely sensitive to slight changes in ICs [26]. The System (2) extremally sensitivity to ICs is verified by simulating the system for 4 seconds, with ICs $(x_0, y_0) = (0.01, 0.01)$ and a small change in these ICs to become $(0.01001, 0.01)$. Fig. 3 shows that System (2) is extremely sensitive to a small change in ICs. For this small change, the system trajectories $(x(t))$ have clearly changed significantly. As a result, System (2) dynamical behaviors are unpredictable and chaotic.

4.3 FRACTAL DIMENSION

The fractal dimension is the fundamental parameter for describing self-similar patterns and processes. Given that it indicates the degree of complexity of natural objects, the concept of fractal dimension is useful in describing them.

Several methods can be considered for calculating fractal dimension [27]. Katz's method, Higuchi's method, the rescaled range method, and Renyi's entropy are a few examples. Higuchi's fractal dimension (HFD) is a time-domain

FIGURE 3. Sensitivity to ICs: $x(t)$ state trajectories with two different ICs

signal complexity metric and a popular nonlinear measure for signal analysis [28]. The following describes Higuchi's method for calculating the fractal dimension of a signal or function [29].

Let $x(1), x(2), \dots, x(N)$ represent an epoch with N signal samples. K is the number of new sub-epochs created from the original epoch and represented by X_m^k , where each sub-epoch is described by the following equation.

$$X_m^k = (x(m), x(m+k), x(m+2k), \dots, x(m + \text{int}((N-k)/k)k)); m = 1, 2, \dots, k; k = 1, 2, \dots, k_{max} \quad (9)$$

In Equation (9), m is the initial time, k is the time interval, $\text{int}(\cdot)$ is the integer real number, and k_{max} is a free parameter. The average length $L_m(k)$ to each of the sub-epochs created can be evaluated as:

$$L_m(k) = \frac{1}{k} \left(\sum_{i=1}^{\text{int}((N-m)/k)} (x(m+ik) - x(m+(i-1)k)) \right) \frac{N-1}{(\text{int}((N-m)/k)k)} \quad (10)$$

Then, the length of epoch $L(k)$ for time interval k is calculated using the mean of the k values as follows:

$$L(k) = \frac{1}{k} \sum_{m=1}^k L_m(k) \quad (11)$$

Finally, HFD can be calculated as follows:

$$HFD = \frac{\ln(L(k))}{\ln(1/k)} \quad (12)$$

In HFD calculation, the selection of free parameter k_{max} is critical. As shown in Fig. 4, HFD values are determined for various k_{max} values with N ranging from 500 to 12,000.

The HFD value of smooth signals (e.g., linear signals) is determined to be 1. The HFD of random white noise signals or complex signals is determined to be ≈ 2 [30]. Given that the fractal dimension is approximated numerically, the computed HFD values may be slightly higher than 2 [31]. As a result, Fig. 4 presents that the constructed System (2) has complex dynamical behaviors, that is, chaotic behaviors.

4.4 AUTOCORRELATION

The autocorrelation metric is another tool for detecting the chaotic dynamical behaviors of systems [32]. In chaotic signals, the correlation coefficient between state variable $x(t)$ for $t < t_1$ and state variable $x(t)$ for $t > t_1$ should be small. That is, low values of the correlation coefficient do not give information on the chaotic features of the time series. The

FIGURE 4. HFD of System (2)

autocorrelation function between the original time series $x(t)$ and the shifted time series $x(t - \tau)$ is described by the following equation [33]. The obtained autocorrelation is illustrated in Fig. 5, which displays that the autocorrelation function quickly drops to zero. Thus, the dynamical behavior of System (2) is unpredictable and chaotic.

$$A(\tau) = \int_0^{\infty} x(t)x(t - \tau)dt \quad (13)$$

FIGURE 5. Autocorrelation for the $x(t)$ state variable of System (2)

4.5 POWER SPECTRUM

A chaotic signal's power spectrum is categorized by its broadband nature [34]. Fig. 6 demonstrates the power spectrum for the $x(t)$ state of System (2). As illustrated, the power spectrum is continuous and broadband. The magnitude power values are also inversely proportional to their frequencies, a common situation in chaotic systems [35].

FIGURE 6. Power spectrum for the $x(t)$ state variable of System (2)

4.6 LYAPUNOV EXPONENT

Lyapunov exponents are determined in nonlinear systems and strongly imply that systems exhibit chaotic behaviors [36]. Lyapunov exponents (Le_i ; $i = 1, 2, \dots, n$) are the numbers that show the exponential attraction or time separation of two adjacent orbits in phase space. Moreover, N Lyapunov exponents exist in an N -dimensional system. Lyapunov exponents determine the dynamical behaviors of systems as follows: one Lyapunov exponent is zero if the system is nonautonomous (e.g., forced pendulum). Chaos is indicated when the system has at least one positive Lyapunov exponent. When the largest Lyapunov exponent is negative, the system trajectories converge in time and the system is unaffected by ICs [37]. If it is positive, then the distance among neighboring trajectories expands exponentially and the system is sensitive to ICs, making it chaotic.

FIGURE 7. Lyapunov exponents of System (2)

As seen in Fig. 7, the Lyapunov exponents of the developed system are investigated numerically, where System (2) parameters are selected as $a = 10$ and $b = 10$ with $(0.01, 0.01)$ ICs. The settling values for Lyapunov exponents in this scenario for a simulation time of 2000 seconds are: $Le_1 = 1.259$ and $Le_2 = -1.408$. As a result, System (2) exhibits a

dynamical chaotic behavior because one of the Lyapunov exponents is positive. Note that the Wolf method is used to find Lyapunov exponents [38]. In addition, the summation of Lyapunov exponents is $Le_1 + Le_2 = -0.149 < 0$. It specifies that System (2) is a dissipative system (i.e., the state trajectories of the system eventually settle into a strange attractor). Furthermore, according to the Kaplan–York estimation, the Lyapunov dimension is a measure of the degree of a chaotic activity in a system (measure the system complexity) [39]. It can be calculated as follows:

$$D_k = J + \frac{\sum_{i=1}^J L_{ei}}{|L_{J+1}|} \quad (14)$$

In Equation (14), J is the largest integer and should be chosen, such that $\sum_{i=1}^J L_{ei} > 0$ and $\sum_{i=1}^{J+1} L_{ei} < 0$. Thus, our developed system can be calculated as follows:

$$D_k = 1 + \frac{L_{e1}}{|L_{e2}|} = 1.894 \quad (15)$$

Given that the obtained Lyapunov dimension is a fraction, it indicates that System (2) exhibits a strange chaotic attractor.

4.7 BIFURCATION DIAGRAM

Bifurcation diagrams are a useful tool for examining system behaviors when one or more system parameters are changed [40]. These diagrams can also be utilized to determine some system characteristics, such as the

FIGURE 8. Bifurcation diagram of the developed system with respect to parameter a

Fig. 8 clearly shows that the chaos route is period-doubling, in which the period of the system's periodic motion is doubled at a certain point of the bifurcation parameter ($a = 9.704$), and then the motion becomes chaotic at a different specific value of the bifurcation parameter ($a = 9.375$). These obtained results suggest that the developed 2D system of nonsmooth ODEs described in Equation (2) is capable of exhibiting complex chaotic dynamical behaviors.

5. CONCLUSIONS

A 2D system of nonsmooth ODEs has been developed in this study. This system can display chaotic dynamical behaviors. Its key dynamical behaviors have been explored, including time-series trajectories, phase pictures of attractors, and equilibria and their stability. Many interesting tools used to detect chaotic behaviors, including chaotic attractor, symmetry, sensitivity to ICs, fractal dimension, autocorrelation, power spectrum, Lyapunov exponent, and bifurcation diagram, have been used to verify the developed system's chaotic behaviors. The ability of the developed 2D nonsmooth system to excite chaotic dynamical behaviors is proven by the numerical simulation results. The discovery of a 2D system of nonsmooth ODEs capable of identifying chaotic dynamical behaviors is a useful research topic. It allows researchers and colleagues to detect chaotic behaviors in any 2D nonsmooth system modeled to represent a real physical system in the real world (e.g., spring-mass systems, epidemic model systems, and predator–prey systems). The main advantage of

the developed system is that it can be used in chaos-based cryptosystems and secure communication systems (e.g., image encryption systems). Such a system contains only two differential equations and exhibits complex dynamics. Therefore, it is simple to be implemented in the real world.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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