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(A-n) – potent operators on Hilbert space

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ABSTRACT: This paper introduced a new class of operators called (A-n)-potent operator on a complex Hilbert space$H$. An operator $T \in B(H)$ is called (A-n)-potent operator if $ATA = T^n$, where n is a positive integer greater than or equal to 2. Some basic properties of such operators were also investigated; the relationship between (A-n)-potent operators and some kinds of operators was also established.

Keywords: Functional analysis, Hilbert space, potent operators.

1. INTRODUCTION

Functional analysis is a part of pure mathematics and plays an increasing role in the applied sciences and mathematics. Its development started about eighty years ago; methods and results of functional analysis are important in various fields of mathematics and applications. One of the important notions in functional analysis is the operator theory investigation which provides solutions to most problems in mathematics [1]. It provides a strong tool to discover solutions to problems in pure mathematics, and excellent techniques to estimate errors between solutions of infinite and finite dimensional problems. It has become a sufficiently large area in response to questions arising in the study of differential and integral equations [2].

Let Hbea Hilbert space and $B(H)$ the algebra of all bounded linear operators on a Hilbert space H. An operator $T \in B(H)$ is called unitary if $UU^* = U^*U = I$, self-adjoint if $T = T^*$, normal if $TT^* = T^*T$ [2], and n-normal if $T^nT^* = T^*T^n$ [3, 4]. The operator $T \in B(H)$ is said to be quasi-normal if $T(T^*T) = (T^*T)T$ [5], quasi n-normal if $T(T^*T^n) = (T^*T^n)T$, and n-power quasi-normal if $T^n(T^*T) = (T^*T)T^n$ [6]. In 2010, Schoutens Hans [7] defined n-potent operator form as: $T^n = T$, and Rijab [8] and [9] defined Triple operator and Triple operator of order n as: An operator $T \in B(H)$ is called (Triple operator) if $(TT^*)T = T(TT^*)$ and (Triple operator of order n) if $(T^nT^*)T = T(T^nT^*)$.

2. SOME PROPERTIES OF (A-N) -POTENT OPERATORS

In this section, we will introduce and study some essential properties of the (A-n)-potent operators.

Definition 2.1. *If $A,\ T \in B(H)$ and n is a positive integer greater than or equal to 2, then, T is called (A-n)-potent operator if $ATA = T^n$.*

Proposition 2.1. *If T is (A-n)- potent operator, then:*

(1) T^* is (A^*-n)- potent operator.

(2) If $A,\ T$ are invertible operators, then, T^{-1} is (A^{-1}-n)- potent operator.

(3) T/M is (A-n)- potent operator.

Proof.

$$(A\ T\ A)^* = A^*T^*A^* = T^{*^n} \tag{1}$$

$$(A\ T\ A)^{-1} = A^{-1}T^{-1}A^{-1} = (T^{-1})^n \tag{2}$$

$$A\ (T/M)\ A = (T^n/M) = \ (T/M)^n \tag{3}$$

Theorem 2.1. *If $A^2 = A$ and T is (A-n)- potent operator, then :*

1- $T^{n+r} = A\ T^{n+r} = T^{n+r}\,A = A\,T^{n+r}\,A \qquad r = 0,1,2,\ldots\ldots$

2- $T^{2n+r} = T^{n+r+1} \qquad r = 0,1,2,\ldots\ldots\ldots\ldots$

Proof.

(1). Since $A\ T\ A = T^n$, then, $A^2\ T\ A = A\ T^n = A\ T^n$
and $A\ TA^2 = T^n\ A$
Since $A^2 = A$, then, $T^n = A\ T\ A = A\ T^n = T^nA$
Now, $T^{n+r} = T^rT^n = T^rT^nA = T^{n+r}A$, and
$T^{n+r} = T^nT^r = A\ T^nT^r = A\ T^{n+r}$, $\qquad r = 0,1,2,\ldots\ldots\ldots$
Hence, $T^{n+r} = A\ T^{n+r} = T^{n+r}A$, $\qquad r = 0,1,2,\ldots\ldots\ldots$
Also, $T^{n+r} = T^{n+r}A = T^{n+r}A^2 = A\ T^{n+r}A$, $\qquad r = 0,1,2,\ldots\ldots\ldots$
Thus, $T^{n+r} = A\ T^{n+r} = T^{n+r}A = A\ T^{n+r}A$, $\qquad r = 0,1,2,\ldots\ldots\ldots$
(2). $T^{2n+r} = T^{n+r}\,T^n = T^{n+r}A\ T\ A = A\ T^{n+r}A$
$= T^{n+r+1}A^2 = T^{n+r+1}A = T^{n+r+1}$, $\qquad r = 0,1,2,\ldots\ldots\ldots$

From theorem (2.1) part (1), we have:

Corollary 2.1. *If $A^2 = A$ and T is (A-n)- potent operator, then, T^{n+r} is (A-1)-potent operator for every $r = 0,1,2,\ldots\ldots\ldots$*
But the converse is not true, as seen in the following example:

Example 2.1. *If* $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and*,

$T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *are operators on Hilbert space C^3.*

Then, $A\ T\ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = T^3 \neq T^2$

Theorem 2.2. *If T and S are commuting (A-n)- potent operators such that $\sum_{k=0}^{n} \binom{n}{k} S^{n-k} T^k = 0$, then, $(T+S)$ is (A-n)- potent operator.*

Proof.

$$A\ (T+S)A = ATA + ASA = T^n + S^n \tag{1}$$

$$(T+S)^n = T^n + \sum_{k=0}^{n} \binom{n}{k} S^{n-k} T^k + S^n = T^n + S^n \tag{2}$$

Then, $(T+S)$ is (A-n)- potent operator.

Theorem 2.3. *If S and T are commuting (A-n)- potent operators such that $A^2 = I$, then, $(S.T)$ is (A-n)- potent operator.*

Proof. Since $A\,SA = S^n$ and $A\,TA = T^n$, then,

$$A\,(S\,T)A = A\ (A\,SA\,A\,TA) = A\,(S^n T^n)A \tag{1}$$

$$(S\,T)^n = (ASA\,ATA\,)^n = (ASTA)^n = A\,(S^n T^n)A \tag{2}$$

Then, $(S.T)$ is (A-n)- potent operator.

The following example shows that if S and T are (A-n)- potent operators, then, it is not necessary that $(S+T)$ and $(S.T)$ are (A-n)- potent operators.

Example 2.2. *If* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *are operators on Hilbert space* C^2, *then,*

$A\,S\,A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} = S^2$ *and* $A\,T\,A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = T^2$

Therefore, S and T are (A-n)- potent operators.

Now, $(S+T) = \begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix}$

$$A\ (S+T)A = \begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} \neq (S+T)^2 = \begin{pmatrix} 7 & 2 \\ -4 & -1 \end{pmatrix}$$

Then, $(S+T)$ is not (A-n)- potent operator.

$$(S.T) = \begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}$$

$$A\ (S.T)A = \begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix} \neq (S.T)^2 = \begin{pmatrix} 2 & 0 \\ -2 & 0 \end{pmatrix}$$

Then, $(S.T)$ is not (A-n)- potent operator.

Proposition 2.2. *Let T be (A-n)-potent operator; then, (αT) is (A-n)-potent operator if $\alpha^n = \alpha$, where α is constant.*

Proof. If $\alpha = 1$, the result is true, and if $\alpha > 1$, then,

$$A\,(\alpha T)A = \ \alpha\,A\,T\,A = \alpha\,T^n \tag{1}$$

$$(\alpha T)^n = \ \alpha^n T^n = \alpha\,T^n \tag{2}$$

$\therefore (\alpha T)$ is (A-n)-potent operator.

3. RELATIONSHIP BETWEEN (A-N - POTENT OPERATOR AND SOME OTHER KINDS OF OPERATORS

Theorem 3.1. *If T is n-potent operator, such that $A\,T = TA$, and $A^2 = I$, then, T is (A-n)- potent operator.*

Proof. Let T be n-potent operator (i.e $T^n = T$);
then, multiply the two-sides by A; we get $A\,T^n A = A\,TA$
$A^2 T^n = ATA \implies T^n = ATA$
$\therefore$ T is (A-n)- potent operator.

The converse is not true, as we saw in example (2.1); T is (A-n)- potent operator but T is not n- potent operator.

Theorem 3.2. *Let T be (A-n)- potent operator such that:*
(a) $A\,T = TA$ (b) $A^2 = I$. *Then$T$:*

(1) If i is n-normal operator, tT en , i is a normal operator.

(2) If T is a Triple operator of order n, then, T is a Triple operator.

(3) If T is n-power quasi-normal operator, then, T is a quasi-normal operator.

Proof.
(1) Since $A\,TA = T^n$ and $T^n T^* = T^* T^n$;
$(A\,T\,A)T^* = T^*(A\,T\,A)$
$(A\,A\,T)T^* = T^*(\,TA\,A)$
$A^2\,T\,T^* = T^*\,T\,A^2 \implies T\,T^* = T^*\,T$
Then, T is a normal operator.
(2) Since A\;TA=T^n, then, {(T}^nT^\ast T=T\;\left(T^nT^\ast\right
$(ATA\,T^*)T = T\,(ATA\,T^*)$
$\left(A^2\,TT^*\right)T = T\left(A^2\,TT^*\right)$
$(T\,T^*)T = T\,(T\,T^*)$ Then, T is a Triple operator.
(3) T^n\left(T^\ast T\right =\left(T^\ast T\right {\;T}^n
$(A\,TA)\,(T^*T) = (T^*T)\,(A\,TA)$
$(A^2T)\,(T^*T) = (T^*T)\,(A^2T)$
$T\,(T^*T) = (T^*T)\,T$
Then, T is a quasi-normal operator.

Proposition 3.1. *If T is (A-n)- potent operator and quasi n-normal operator such that* (1) $A\,T = TA$ (2) $A\,T^* = T^*A$, *then :*
T Is n-power quasi-normal operator

Proof. Since $A\,TA = T^n$ and $T\,(T^*T^n) = (T^*T^n)\,T$, then,
$T\,(T^*ATA) = (T^*ATA)\,T$
$(A\,T\,T^*\,T\,A) = (T^*\,T\,A\,T\,A)$
$(A\,T\,A\,T^*\,T) = (T^*\,T\,A\,T\,A)$
$T^n\,(T^*\,T) = (T^*\,T)T^n$ is n-power quasi-normal operator .

Remark 3.1.

(1) If T is an idempotent operator, then, T is (A-n)- potent operator.

(2) If T is a self-adjoint operator, then, T is (A-n - potent operator

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