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On the Involutive Matrices of the k^{th} Degree

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ABSTRACT: In this study, the gradation of involutive matrices, whose definitions were given before, is conducted. The solutions of the equation $x^2 = 1$ in real numbers are ± 1. Meanwhile, in the solution of the equation $x^k = 1$ in real numbers, there is always the number ± 1 that is independent of the power of $k \in \mathbb{Z}^+$. This feature, which is revealed by this equation in real numbers, is the subject of the research. In particular, the kind of situation in which the equation would display in the matrices is determined. Initially, the second-order square matrices are studied by obtaining some of their properties. Then, new cases arising from the known addition, subtraction, multiplication, scalar multiplication, and division operations of this set of second-order and quadratic involutive matrices are investigated. Some properties of third- and second-degree involutive matrices, which can be provided and exist, are emphasized. Comparisons are performed on the involutive matrices. Examples, theorems, and lemmas emerging between these two types of matrices are given. New concepts are introduced to the literature by using linear matrix equations and multiplications by the matrices.

Keywords: matrix theory, involutive, periodic, involutive matrix, periodic matrix, degree.

AMS Subject Classification: 08A40, 15A09, 15A15, 15A80, 11K70, 70KC42, 15B36

1. INTRODUCTION

The solution of the equation $x^2 = 1$ in real numbers is $x = \pm 1$. Meanwhile, the solutions of $x^k = 1$ are ± 1 for any integer k.

Let F be a field and $M_n(F) = \left\{ \left[a_{ij} \right]_n \mid a_{ij} \in F, n \in \mathbb{Z}^+ \right\}$. Here, $M_n(F)$ is regarded as regular matrices. The equation $AX = B$ is written for the matrices $A, X, B \in M_n(\mathbb{R})$. The matrix equation under these conditions always has a solution of the form $X = \frac{B}{A}$ [1–4].

The equation $X^2 = I_n$ is obtained when $A = X$ and $B = I_n$ from the equation $AX = B$. Necessary conditions in the solution of this equation indicate the following definition [5–7]:

$$X^2 = I \Rightarrow X^2 - I_n = [0] \Rightarrow X = I_n \vee X = -I_n.$$

Moreover, the solutions of the equation $X^2 - I_n = [0]$ are zero-dividing matrices [8, 9].

$$(X + I_n)(X - I_n) = [0] \Rightarrow X + I_n \neq [0] \vee X - I_n \neq [0].$$

In mathematics, an involutory matrix is a square matrix that is its own inverse. That is, multiplication by matrix A is an involution if and only if $A^2 = I_n$, where I_n is the $n \times n$ involutive matrix. Involutory matrices are all square roots of the identity matrix. This scenario is simply a consequence of an instance in which any non-singular matrix multiplied by its inverse is the identity [10–13].

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2. INVOLUTIVE MATRICES OF THE k^{th} DEGREE

While studying prime matrices, I came to the conclusion that known involutive matrices may involve degrees. The generalized definition of involution is given below [2, 4, 10–12, 14]. The generalized definition of involutive is given below.

Definition 1. *A $n \times n$ matrix A is called k^{th}-degree involutive matrices if and only if $A^k = I_n$, where $k \geq 2$. The smallest k positive integer satisfying this condition is called degree of involutivity.*

The zero and unit matrices have no remarkable properties. The set of involutive matrices of a field F is denoted by $N^k_{M_n}(F)$. In this study, $F = \mathbb{R}$.

$$N^k_{M_n}(\mathbb{R}) = \left\{A = \left[a_{ij}\right]_n \mid A^k = I_n, \det(A) \neq 0, a_{ij} \in \mathbb{R}\right\}$$

Example 1. *Let* $A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$ *be a second-order matrix. Matrix A is a second-order 2^{nd} degree involutive matrix because*

$$A^2 = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example 2. *Let matrix A be given by* $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. *Matrix A is the $6^{th}-$ degree involutive matrix. Hence,*

$$A^6 = A^5 A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Definition 2. *Definition 2.2. A square matrix A satisfying the relation $A^{k+1} = A$ for some positive integer k is called a periodic matrix. The period of the matrix represents the least value of* k *for which $A^{k+1} = A$ holds [7].*

Lemma 1. *Let $A \in N^k_{M_n}(\mathbb{R})$ be a matrix. Then, the period of matrix A is k.*

Proof. As matrix A is involutive of the k^{thl} order, $A^k = I_n$.

$$A^{k+1} = A, \text{ where } k \in \mathbb{Z}^+$$

Lemma 2. *Let A be an n order and $k^{th}-$ degree involutive matrix. Then, matrix A^{-1} is a $k^{th}-$ degree involutive matrix.*

Proof. As matrix A is involutive of the k^{th} degree, then $A^k = I_n$.

$$\left(A^{-1}\right)^k = \left(A^k\right)^{-1} = I_n, \text{ where } k \in \mathbb{Z}^+$$

When $k = 1$, then $AA = A$, which is called an idempotent matrix.

Remark 1. *Each idempotent matrix is a periodic matrix with period 1 [14].*

Lemma 3. *Let $A \in N^2_{M_2}(\mathbb{R})$. The elements of this matrix should satisfy the following conditions:*

$$\begin{array}{l} i.\ |a_{11}| = \sqrt{1 - 4a_{12}a_{21}} = |a_{22}| \wedge a_{11}^2 + a_{12}a_{21} = a_{22} \\ ii.\ a_{12} = 0, a_{21} = 0 \vee a_{11} = -a_{22} \\ iii.\ \begin{array}{c} a_{21} \neq 0 \\ V \\ a_{21} \neq 0 \end{array} \Rightarrow \begin{cases} a_{21} > 0 \Rightarrow \frac{1}{4a_{21}} > a_{12} \\ a_{21} < 0 \Rightarrow \frac{1}{4a_{21}} < a_{12} \end{cases} \end{array}$$

If $a_{21} = 0$, then

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. If matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is the second-degree involutive matrix, then $A^2 = I_2$.

$$A^2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a_{11}^2 + a_{12}a_{21} - 1 = 0 \tag{1}$$
$$a_{11}a_{12} + a_{12}a_{22} = 0 \tag{2}$$
$$a_{11}a_{21} + a_{21}a_{22} = 0 \tag{3}$$
$$a_{22}^2 + a_{12}a_{21} - 1 = 0 \tag{4}$$

The equation $|a_{11}| = \sqrt{1 - 4a_{12}a_{21}} = |a_{22}|$ is easily obtained by equations (1) and (4).

$\begin{matrix} a_{21} \neq 0 \\ \vee \\ a_{21} \neq 0 \end{matrix} \Rightarrow \begin{cases} a_{21} > 0 \Rightarrow \frac{1}{4a_{21}} > a_{12} \\ a_{21} < 0 \Rightarrow \frac{1}{4a_{21}} < a_{12} \end{cases}$ is easily obtained by equations (2) and (3).

If $a_{12} = 0 \vee a_{21} = 0$, then $|a_{11}| = 1, |a_{22}| = 1$.

Property. Let A be a regular second-order square matrix such that lemma 2.3 is satisfied. Then, the matrix $A = \begin{bmatrix} \frac{a_{11}}{\sqrt{a_{11}^2+a_{12}a_{21}}} & \frac{a_{12}}{\sqrt{a_{11}^2+a_{12}a_{21}}} \\ \frac{a_{21}}{\sqrt{a_{11}^2+a_{12}a_{21}}} & \frac{-a_{11}}{\sqrt{a_{11}^2+a_{12}a_{21}}} \end{bmatrix}$ is a second-degree involutive matrix.

Example 3. *$A = \begin{bmatrix} \frac{3}{\sqrt{7}} & \frac{-8}{4\sqrt{7}} \\ \frac{1}{4\sqrt{7}} & -\frac{3}{\sqrt{7}} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{\sqrt{13}} & -\frac{1}{\sqrt{13}} \\ \frac{-16}{\sqrt{13}} & -\frac{4}{\sqrt{13}} \end{bmatrix}$ matrices are second-degree involutive matrices. Therefore, $a_{11} = 3, a_{22} = -3, a_{12} = -8, a_{21} = \frac{1}{4}$ and $a_{11}^2 + a_{12}a_{21} = 7$ for the matrix $\begin{bmatrix} 3 & -8 \\ \frac{1}{4} & -3 \end{bmatrix}$.*

$$A^2 = \begin{bmatrix} \frac{3}{\sqrt{7}} & \frac{-8}{4\sqrt{7}} \\ \frac{1}{4\sqrt{7}} & -\frac{3}{\sqrt{7}} \end{bmatrix}\begin{bmatrix} \frac{3}{\sqrt{7}} & \frac{-8}{4\sqrt{7}} \\ \frac{1}{4\sqrt{7}} & -\frac{3}{\sqrt{7}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a_{11} = 2, a_{22} = -2, a_{12} = -\frac{1}{4}, a_{21} = 3$ and $a_{11}^2 + a_{12}a_{21} = \frac{13}{4}$ for $\begin{bmatrix} 2 & -\frac{1}{4} \\ 3 & -2 \end{bmatrix}$

$$B^2 = \begin{bmatrix} \frac{4}{\sqrt{13}} & -\frac{1}{2\sqrt{13}} \\ \frac{-6}{\sqrt{13}} & -\frac{4}{\sqrt{13}} \end{bmatrix}\begin{bmatrix} \frac{4}{\sqrt{13}} & -\frac{1}{2\sqrt{13}} \\ \frac{-6}{\sqrt{13}} & -\frac{4}{\sqrt{13}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lemma 4. *Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a second-order involutive matrix in $N_{M_2}^3(\mathbb{R})$. The elements of this matrix should satisfy the following conditions:*

i. $a_{21}(2a_{11} + a_{22}) \neq 0 \Rightarrow a_{12} = \frac{1-a_{11}^3}{2a_{11}a_{21}+a_{21}a_{22}}$

ii. $a_{12}(2a_{22} + a_{11}) \neq 0 \Rightarrow a_{21} = \frac{1-a_{22}^3}{2a_{12}a_{22}+a_{11}a_{12}}$

iii. $a_{11} = -\frac{a_{22}}{2} \mp \frac{1}{2}\sqrt{-3a_{22}^2 - 4a_{12}a_{21}} = a_{22}$

Proof. If matrix A is the third-degree involutive matrix, then $A^3 = I_2$.

$$A^3 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$a_{11} = -\frac{a_{22}}{2} \pm \frac{1}{2}\sqrt{-3a_{22}^2 - 4a_{12}a_{21}}, \text{ where } a_{12} \neq 0 \vee a_{21} \neq 0$$

$$a_{12} = \frac{1 - a_{11}^3}{2a_{11}a_{21} + a_{21}a_{22}}, 2a_{11}a_{21} + a_{21}a_{22} \neq 0$$

$$a_{21} = \frac{1 - a_{22}^3}{2a_{12}a_{22} + a_{11}a_{12}}, 2a_{12}a_{22} + a_{11}a_{12} \neq 0$$

Example 4. *Conditions (i) to (iii) are fulfilled for* $a_{11} = 0, a_{22} = -1$ *and* $a_{12} = -1, a_{21} = 1$. *Matrix A is a thirddegree involutive matrix because*

Matrix A is not a quadratic identity matrix.

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However, matrix A is a third-order involutive matrix.

Lemma 5. *Let* $A \in N_{M_n}^k(\mathbb{R})$. *Then,*

$$A^{kt+1} = I_n, \text{ where } t \geq k, t \in \mathbb{Z}^+.$$

Proof. The proof of Lemma 2.5 is determined according to Lemma 1. For k,

and

$$A^k = I_n.$$

$$A^{kt+1} = \left(A^k\right)^t A = A$$

Theorem 1. *Let* $A \in M_n(\mathbb{R})$. *Then, A is a k periodic matrix if and only if A is a* $k-$degree *involutive matrix [1, 2,* 13, 14]

Proof. $\Rightarrow$ *: If the* $A \in M_n(\mathbb{R})$ *regular matrix is a k periodic matrix, then*

:$\Leftarrow$ *If A is k-degree involutive matrix, then,*

$$A^{k+1} = A \Rightarrow A^k = I_n.$$

$$A^k = I_n \Rightarrow A^{k+1} = A.$$

3. CONCLUSIONS AND DISCUSSIONS

The solution of systems of linear matrix equations indicates that the concept of multipliers in the products of matrices contain more information. This development offers new definitions and theorems to the literature. Furthermore, it enables the introduction of new properties, general definitions, and theorems of known definitions in the literature. Related discussions are expected to lead to new research.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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