The Dynamics of a Single Species in a Polluted Environment

Raid Kamel Naji^{1*}, Mona Ghassan Younis², Mohammad Naeemullah³

¹Department of Mathematics, College of Science, University of Baghdad, Baghdad, IRAQ

²Department of computer science, college of education, AL-Iraqia University, Iraq

³Department of Computer Science Maulana Azad College. Rauza Bagh Aurangabad. Maharashtra, INDIA

*Corresponding Author: Raid Kamel Naji

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ABSTRACT: This article proposed and analysed a nonlinear mathematical model that consist of a single species in a polluted environment (PE). The proposed model was also discussed in terms of its uniqueness, existence, and boundedness of the solution. Also, each possible equilibrium point was analysed for local stability, followed by investigation of the global dynamics of the system using the Lyapunov functions. The effects of the presence of toxicants on the dynamics of a single species in the PE was numerically investigated.

Keywords: Prey-predators; Toxic; Environment.

1. INTRODUCTION

The effect of pollutants on the environment and ecological communities has received much attention from the conservation and environmental perspectives. Many studies have focused on the impacts of several pollutants, such as oil spill, forest degradation, and dumping of toxic materials into water bodies [1–3]. The effect of pollutants on the population of a single species has been well studied; for instance, the study by Hallam et al. [4] focused on the impact of the presence of pollutants in environments hosting single species; the study assumed that the rate of growth of such single species tend to decrease linearly as the concentration of the pollutant increases while the associated carrying capacity is independent of the pollutant concentration in the environment. The models of the impact of pollutants on a single species and predator-prey systems have been proposed by Freedman & Shukla [5]; the study assumed a significant decline in the growth rate of the species at higher rates of pollutants intake while the carrying capacity is significantly reduced at high pollutants concentration in the environment. [6–8]

In this chapter, however, a nonlinear mathematical model consisting of a single species population living in a polluted environment (PE) was proposed and analysed. It is assumed that a single species population is directly affected by the pollutants formed by precursor of species and indirectly affected by the pollutants released into the environment from external sources. The proposed model was analytically and numerically investigated for its dynamic behaviour. [9, 10]

2. MATHEMATICAL MODEL

For any single species living in a PE, the dynamics of the system can be described as follows:

$$\frac{dx}{dt} = rx - ax^2 - \alpha xZ \quad (1a)$$

$$\frac{dZ}{dt} = kxP - \gamma xZ - d_1Z + \delta d_2P \quad (1b)$$

$$\frac{dP}{dt} = Q - d_2P - kxP + \vartheta d_1Z \quad (1c)$$

where $x(t)$ represents the density of the species at time t , $Z(t)$ represents the toxicant (pollutant) concentration in species $x(t)$ at time t , and $P(t)$ is the toxicant concentration presented in the environment at time t . The parameters $r, a, \alpha, k, \gamma, Q$ and $d_i (i = 1, 2)$ are positive constants, while δ & ϑ are non-negative constants. Furthermore, the constant r represents the rate of intrinsic growth of species $x(t)$ in a pollutant-free environment; α represent the coefficient of the rate of depletion of $x(t)$ due to the presence of the pollutant; a is the intra-specific competition within species $x(t)$; Q represents the rate of exogenous entrance of the pollutant into the considered environment; k is the rate of pollutant depletion in the environment as it is taken up by the species; d_1 is the coefficient of the natural rate of depletion of $Z(t)$ as it is taken up by the species, with ϑ ($0 \leq \vartheta < 1$) of d_1Z that may again enter the environment; d_2 represents the washout rate of the pollutant in the system due to some natural processes, such as biotransformation, while δ ($0 \leq \delta < 1$) is the fraction of d_2P which may be uptake by the species; finally, γ is the depletion rate coefficient of $Z(t)$ due to its removal by species $x(t)$.

Clearly, due to the biological meaning of the dependent variables $x(t)$, $Z(t)$ and $P(t)$ of system (2.1), the domain of the system can be written as $\mathfrak{R}_+^3 = \{(x, Z, P) \in \mathfrak{R}^3 : x \geq 0; Z \geq 0; P \geq 0\}$. Furthermore, it is easy to verify that, system (1) is continuous and has a continuous partial derivative in $\mathfrak{R}_+^3$. Hence for each set of initial conditions, say $x(0) \geq 0, Z(0) \geq 0$ and $P(0) \geq 0$, system (1) has a specialized solution. Furthermore, the attraction region for the solutions of system (1) can be determined using the following Lemma.

Lemma 1. The set $\Omega = \{(x, Z, P) \in \mathfrak{R}_+^3 : 0 \leq x \leq \frac{r}{a}; Z_m \leq Z(t); P_m \leq P(t); Z + P \leq \frac{Q}{M}\}$ where $P_m = \frac{aQ}{kr + d_2a}$,

$Z_m = \frac{\delta d_2 a^2 Q}{(kr + d_2 a)(\gamma r + d_1 a)}$ and $M = \min\{d_1(1 - \vartheta), d_2(1 - \delta)\}$ represents the region of attraction for all solution of system (1) initiate in $\mathfrak{R}_+^3$.

Proof. From equation (1a), we have:

$$\frac{dx}{dt} \leq rx \left(1 - \frac{ax}{r}\right)$$

So, according to the comparison theorem [2], it is easy to see that:

$$\lim_{t \rightarrow \infty} \text{Sup } x(t) \leq \frac{r}{a} \Rightarrow x(t) \leq \frac{r}{a} \forall t > 0 \quad (2)$$

Now, from equations (2.1c) and (2.2), it is observed that:

$$\frac{dP}{dt} \geq Q - \left(d_2 + k\frac{r}{a}\right)P \Leftrightarrow \frac{dP}{dt} + \left(d_2 + k\frac{r}{a}\right)P \geq Q$$

Again, the comparison theorem gives that:

$$\liminf_{t \rightarrow \infty} P(t) \geq \frac{aQ}{kr + d_2a} = P_m \quad (3)$$

Also, from equations (1b) with (2) and (3), it is observed that:

$$\frac{dZ}{dt} + \left(\infty\frac{r}{a} + d_1\right)Z \geq \frac{\delta d_2 a Q}{kr + d_2 a}$$

Hence, we get:

$$\lim_{t \rightarrow \infty} \inf Z(t) \geq \frac{\delta d_2 a^2 Q}{(\gamma r + d_1 a)(kr + d_2 a)} = Z_m \quad (4)$$

Now, let us define that $B(t) = Z(t) + P(t)$. Hence,

$$\frac{dB}{dt} = \frac{dZ}{dt} + \frac{dP}{dt} \leq -d_1(1 - \vartheta)Z - d_2(1 - \delta)P + Q$$

Therefore,

$$\frac{dB}{dt} \leq Q - M(B) \Leftrightarrow \frac{dB}{dt} + M(B) \leq Q \quad (5)$$

Clearly, as $t \rightarrow \infty$, it is easy to verify that $\text{Sup } B \rightarrow \frac{Q}{M}$. Hence $B(t) \leq \frac{Q}{M}, \forall t > 0$, and then, the proof is complete.

3. EQUILIBRIA AND LOCAL STABILITY ANALYSIS

This section discussed the analysis of the existence and local stability of system (1); the systems is observed to have a maximum of 2 non-negative equilibrium points, say $e_1 = (0, \tilde{Z}, \tilde{P})$ and $e_2 = (\bar{x}, \bar{Z}, \bar{P})$.

The existence of equilibrium point e_1 in $\mathfrak{R}_+^3$ only holds if the following equations have a positive solution:

$$-d_1 Z + \delta d_2 P = 0 \quad (6a)$$

$$Q - d_2 P + \vartheta d_1 Z = 0 \quad (6b)$$

Using a little algebraic manipulation gives that:

$$\tilde{Z} = \frac{\delta Q}{d_1(1 - \vartheta \delta)} \quad (7a)$$

$$\tilde{P} = \frac{Q}{d_2(1 - \vartheta \delta)} \quad (7b)$$

Clearly, $\tilde{Z}$ and $\tilde{P}$ are positive, provided that $\vartheta \delta < 1$ which is always satisfied because ϑ and $\delta \in (0, 1)$.

However, the equilibrium point e_2 exist in $\text{Int. } \mathfrak{R}_+^3$ if the following set of equations have a positive solution.

$$r - ax - \alpha Z = 0 \quad (8a)$$

$$kxP - \gamma xZ - d_1 Z + \delta d_2 P = 0 \quad (8b)$$

$$Q - d_2 P - kxP + \vartheta d_1 Z = 0 \quad (8c)$$

A straightforward computation shows that:

$$\bar{Z} = \frac{Q(kx + \delta d_2)}{d_1 d_2(1 - \vartheta \delta) + d_1 k(1 - \vartheta) + \gamma d_2 x + \gamma k x^2} \quad (9a)$$

$$\bar{P} = \frac{Q(d_1 + \gamma x)}{d_1 d_2(1 - \vartheta \delta) + d_1 k(1 - \vartheta) + \gamma d_2 x + \gamma k x^2} \quad (9b)$$

While $\bar{x}$ is a positive root of the following third-order equation:

$$\begin{aligned} & ak\gamma x^3 + [a(d_1 k(1 - \vartheta) + d_2 \gamma) - rk\gamma]x^2 \\ & + [ad_1 d_2(1 - \vartheta \delta) + \gamma kQ - r(d_1 k(1 - \vartheta) + d_2 \gamma)]x \\ & + \gamma \delta d_2 Q - rd_1 d_2(1 - \vartheta \delta) = 0 \end{aligned} \quad (10)$$

Consequently, by using the Descartes rule, equation (10) has a unique positive root if and only if:

$$a(d_2 \gamma + d_1 k(1 - \vartheta)) > rk\gamma \text{ and } rd_1(1 - \vartheta \delta) > \gamma \delta Q \quad (11a)$$

Or

$$\gamma \delta Q < rd_1(1 - \vartheta \delta) \text{ and } \frac{ad_1 d_2(1 - \vartheta \delta) + \gamma kQ}{r} < d_1 k(1 - \vartheta) + d_2 \gamma < \frac{rk\gamma}{a} \quad (11b)$$

Therefore $e_2 = (\bar{x}, \bar{Z}, \bar{P})$ uniquely exists in $\text{Int. } \mathfrak{R}_+^3$ if only condition (3.6a) or (3.6b) hold.

Now, system (1) can be studied for local stability near the established equilibrium points above by first computing the Jacobian matrix of the system at each of the established equilibrium points, followed by the derivation of the eigenvalues.

The Jacobian matrix of system (1) at e_1 is computed as follows:

$$J(e_1) = \begin{bmatrix} r - \alpha \tilde{Z} & 0 & 0 \\ k\tilde{P} - \gamma \tilde{Z} & -d_1 & \delta d_2 \\ -k\tilde{P} & \vartheta d_1 & -d_2 \end{bmatrix}$$

Clearly, this Jacobian matrix has the following characteristic equation:

$$(r - \alpha \tilde{Z} - \lambda)(\lambda^2 + (d_1 + d_2)\lambda + d_1 d_2(1 - \vartheta \delta)) = 0$$

Therefore, either

$$r - \alpha\tilde{Z} - \lambda = 0 \quad (12a)$$

Or

$$\lambda^2 + (d_1 + d_2)\lambda + d_1d_2(1 - \vartheta\delta) = 0 \quad (12b)$$

Hence, from equation (12a) we obtain:

$$\lambda_x = r - \alpha\tilde{Z} \quad (13a)$$

Here, λ_x denotes the eigenvalue in the x -direction. On the other hand, from equation (12b), we obtain:

$$\begin{aligned} \lambda_Z + \lambda_P &= T = -(d_1 + d_2) < 0 \\ \lambda_Z \cdot \lambda_P &= D = d_1d_2(1 - \vartheta\delta) > 0 \end{aligned} \quad (13b)$$

Hence, by using Trace-Determinant criterion, equation (12b) has roots (eigenvalues say λ_Z and λ_P) with negative real parts.

Thus, the equilibrium point e_1 is local & stable asymptotically in $\mathfrak{R}_+^3$ if and only if:

$$r < \alpha\tilde{Z} \Leftrightarrow rd_1(1 - \vartheta\delta) < \alpha\delta Q \quad (14a)$$

However, the equilibrium point e_1 is considered an unstable saddle point with stable and local manifold in the ZP -plane and with local and unstable manifold in the x -direction, provided that:

$$r > \alpha\tilde{Z} \Leftrightarrow rd_1(1 - \vartheta\delta) > \alpha\delta Q \quad (14b)$$

The Jacobian matrix of system (1) at the equilibrium point e_2 , say $J(e_2)$, can be written as:

$$J(e_2) = \begin{bmatrix} r - 2a\bar{x} - \alpha\tilde{Z} & -\alpha\bar{x} & 0 \\ k\bar{P} - \gamma\tilde{Z} & -\gamma\bar{x} - d_1 & k\bar{x} + \delta d_2 \\ -k\bar{P} & \vartheta d_1 & -d_2 - k\bar{x} \end{bmatrix}$$

Therefore, the characteristic equation of $J(e_2)$ can be written as:

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \quad (15)$$

where,

$$\begin{aligned} A_1 &= -(r - 2a\bar{x} - \alpha\tilde{Z}) + (\gamma + k)\bar{x} + d_1 + d_2 \\ A_2 &= -(r - 2a\bar{x} - \alpha\tilde{Z})((\gamma + k)\bar{x} + d_1 + d_2) + \alpha\bar{x}(k\bar{P} - \gamma\tilde{Z}) \\ &\quad + \gamma\bar{x}(d_2 + k\bar{x}) + d_1d_2(1 - \vartheta\delta) + d_1k\bar{x}(1 - \vartheta) \\ A_3 &= -(r - 2a\bar{x} - \alpha\tilde{Z})[\gamma\bar{x}(d_2 + k\bar{x}) + d_1d_2(1 - \vartheta\delta) + d_1k\bar{x}(1 - \vartheta)] \\ &\quad - \alpha\bar{x}[\gamma\tilde{Z}(d_2 + k\bar{x}) - d_2k\bar{P}(1 - \delta)] \end{aligned}$$

So, based on the Routh-Hurwitz criterion, it is clear that equation (15) has roots with negative real parts when $A_i > 0 \quad \forall i = 1, 3$ and $\Delta = A_1A_2 - A_3 > 0$.

A straightforward computation shows that A_3 and Δ can be written as:

$$\begin{aligned} A_3 &= -\gamma\bar{x}(d_2 + k\bar{x})(r - 2a\bar{x}) - (r - 2a\bar{x} - \alpha\tilde{Z})[d_1d_2(1 - \vartheta\delta) + d_1k\bar{x}(1 - \vartheta)] \\ &\quad + \alpha d_2 k \bar{x} \bar{P} (1 - \delta) \\ \Delta &= -[r - 2a\bar{x} - \alpha\tilde{Z} - (\gamma\bar{x} + d_1)][-(r - 2a\bar{x})(\gamma + d_1) + \alpha(d_1\tilde{Z} + k\bar{x}\bar{P})] \\ &\quad + [(\gamma + k)\bar{x} + d_1 + d_2][\gamma\bar{x}(d_2 + k\bar{x}) + d_1d_2(1 - \vartheta\delta) + d_1k\bar{x}(1 - \vartheta)] \\ &\quad + (r - 2a\bar{x} - \alpha\tilde{Z})(d_2 + k\bar{x})[r - 2a\bar{x} - \alpha\tilde{Z} - 2\gamma\bar{x} - 2d_1 - d_2 - k\bar{x}] \\ &\quad + \alpha k^2 \bar{x}^2 \bar{P} + \alpha k d_2 \bar{x} \bar{P} \end{aligned}$$

Therefore, $A_i > 0 \quad \forall i = 1, 3$ and $\Delta > 0$ provided that the following condition holds:

$$r < 2a\bar{x} \quad (16)$$

Then, the equilibrium point e_2 is locally and asymptotically stable.

4. GLOBAL STABILITY ANALYSIS

This section presents the analytical analysis of the global stability of system (1) using the Lyapunov method. The global stability for the established equilibrium points e_1 and e_2 can be proven using the following theorems:

Theorem 1. *If the following conditions hold:*

$$\sqrt{\frac{r}{\alpha}} \tilde{Z} < \frac{\delta d_2 a^2 Q}{(d_2 a + kr)(\gamma r + d_1 a)} \quad (17a)$$

$$\frac{Q}{M} < \frac{\tilde{Z}}{\tilde{P}} \left(\frac{aQ}{d_2 a + kr} \right) \quad (17b)$$

$$\left(\delta d_2 + \frac{\tilde{Z}}{\tilde{P}} \vartheta d_1 \right)^2 < 4 d_1 d_2 \frac{\tilde{Z}}{\tilde{P}} \quad (17c)$$

Then, the equilibrium point, e_1 of system (1) is globally and asymptotically stable in Ω .

Proof. Consider the following function:

$$L_1(x, Z, P) = \frac{\gamma \tilde{Z}}{\alpha} x + \frac{(Z - \tilde{Z})^2}{2} + \frac{\tilde{Z}}{\tilde{P}} \frac{(P - \tilde{P})^2}{2}$$

Clearly, $L_1 : \mathfrak{R}_+^3 \rightarrow \mathfrak{R}$ is a definite positive function which upon differentiation with respect to time t , gives that:

$$\frac{dL_1}{dt} = \frac{\gamma \tilde{Z}}{\alpha} \frac{dx}{dt} + (Z - \tilde{Z}) \frac{dZ}{dt} + \frac{\tilde{Z}}{\tilde{P}} (P - \tilde{P}) \frac{dP}{dt}$$

Substituting the value of $\frac{dx}{dt}$, $\frac{dZ}{dt}$ and $\frac{dP}{dt}$ from system (1) in this equation, followed by some algebraic manipulations, gives that:

$$\begin{aligned} \frac{dL_1}{dt} = & - \left[\gamma Z^2 - \frac{\gamma r}{\alpha} \tilde{Z} \right] x - \left[\frac{\tilde{Z}}{\tilde{P}} P - Z \right] \left[kxP - \frac{\gamma \tilde{Z}}{\alpha} ax^2 - d_1 (Z - \tilde{Z})^2 \right. \\ & \left. + \left[\delta d_2 + \frac{\tilde{Z}}{\tilde{P}} \vartheta d_1 \right] (Z - \tilde{Z})(P - \tilde{P}) - \frac{\tilde{Z}}{\tilde{P}} d_2 (P - \tilde{P})^2 \right] \end{aligned}$$

Now, it is easy to verify that, condition (17a) guarantees that $\gamma Z^2 - \frac{\gamma r}{\alpha} \tilde{Z} > 0$, while condition (17b) guarantees that $\frac{\tilde{Z}}{\tilde{P}} P - Z > 0$. Finally, condition (17c) guarantees the possibility of writing the last three terms in quadratic form: therefore:

$$\frac{dL_1}{dt} < - \left[(d_1)^{\frac{1}{2}} (Z - \tilde{Z}) - \left(\frac{\tilde{Z}}{\tilde{P}} \right)^{\frac{1}{2}} (P - \tilde{P}) \right]^2$$

So; $\frac{dL_1}{dt}$ is negative definite and hence, L_1 is a Lyapunov function. Therefore, e_1 is globally and asymptotically stable, thereby completing the proof.

Theorem 2. *If the following inequalities hold:*

$$\left[\frac{kaQ}{ad_2 + kr} - \alpha - \frac{\gamma Q}{M} \right]^2 < a(d_1 + \gamma \bar{x}) \quad (18a)$$

$$\left[k \frac{Q}{M} \right]^2 < \frac{a(k\bar{x} + d_2)}{4} \quad (18b)$$

$$(k\bar{x} + \delta d_2 + \vartheta d_1)^2 < \frac{(d_1 + \gamma \bar{x})(k\bar{x} + d_2)}{4} \quad (18c)$$

Then, the positive equilibrium point e_2 is global and asymptotically stable in Ω .

Proof. Assume that:

$$L_2(x, Z, P) = \left(x - \bar{x} - \bar{x} \ln \frac{x}{\bar{x}} \right) + \frac{(Z - \bar{Z})^2}{2} + \frac{(P - \bar{P})^2}{2}$$

Clearly $L_2 : \mathfrak{R}_+^3 \rightarrow \mathfrak{R}$ is a positive definite function which upon differentiation with respect to time t , gives:

$$\frac{dL_2}{dt} = \frac{(x - \bar{x})}{x} \frac{dx}{dt} + (Z - \bar{Z}) \frac{dZ}{dt} + (P - \bar{P}) \frac{dP}{dt}$$

Substituting the value of $\frac{dx}{dt}$, $\frac{dZ}{dt}$ and $\frac{dP}{dt}$ from system (1) in this equation, followed by some algebraic manipulations, gives the following:

$$\begin{aligned} \frac{dL_2}{dt} = & -\frac{a}{2}(x - \bar{x})^2 + [kP - \alpha - \gamma Z](x - \bar{x})(Z - \bar{Z}) - \frac{d_1 + \gamma \bar{x}}{2}(Z - \bar{Z})^2 \\ & -\frac{a}{2}(x - \bar{x})^2 - kP(x - \bar{x})(P - \bar{P}) - \frac{k\bar{x} + d_2}{2}(P - \bar{P})^2 \\ & -\frac{d_1 + \gamma \bar{x}}{2}(Z - \bar{Z})^2 + (k\bar{x} + \delta d_2 + \vartheta d_1)(Z - \bar{Z})(P - \bar{P}) - \frac{k\bar{x} + d_2}{2}(P - \bar{P})^2 \end{aligned}$$

Since $[kP - \alpha - \gamma Z]^2 < \left[\frac{kaQ}{ad_2 + kr} - \alpha - \frac{\gamma Q}{M} \right]^2$

Now, it is easy to verify that, under the given conditions, we can rewrite the above equation as a sum of quadratics. Then;

$$\begin{aligned} \frac{dL_2}{dt} < & - \left[\left(\frac{a}{2} \right)^{\frac{1}{2}} (x - \bar{x}) - \left(\frac{d_1 + \gamma \bar{x}}{2} \right)^{\frac{1}{2}} (Z - \bar{Z}) \right]^2 \\ & - \left[\left(\frac{a}{2} \right)^{\frac{1}{2}} (x - \bar{x}) - \left(\frac{k\bar{x} + d_2}{2} \right)^{\frac{1}{2}} (P - \bar{P}) \right]^2 \\ & - \left[\left(\frac{d_1 + \gamma \bar{x}}{2} \right)^{\frac{1}{2}} (Z - \bar{Z}) - \left(\frac{k\bar{x} + d_2}{2} \right)^{\frac{1}{2}} (P - \bar{P}) \right]^2 \end{aligned}$$

So; $\frac{dL_2}{dt}$ is negative definite and hence L_2 is a Lyapunov function. Hence e_2 is a globally asymptotically stable and then the proof is complete.

5. NUMERICAL SIMULATION

This section presents the numerical analysis of the global dynamics of system (1) for different parameters and initial conditions based on the predictor-corrector approach using the six-order Runge-Kutta technique [7]; the paths of system (1) are then drawn.

For the following set of parameters:

$$\begin{aligned} r = 1, a = 0.1, k = 1, d_1 = 1, d_2 = 1, \alpha = 0.25, \gamma = 0.3 \\ \vartheta = 0.4, \delta = 0.3, Q = 3 \end{aligned} \quad (19)$$

Figure 1 presents the attractive sets of system (1) for varying sets of initial conditions.

It's observed that system (1) approaches asymptotically to the globally stable equilibrium point $e_2 = (7.64, 0.94, 0.39)$ for the set of parameters given by equation (19).

Now, to investigate the effect of the intrinsic growth rate for the species x in the existence of toxicant (i.e., parameter r), system (1) is numerically solved for different values of $r = 0.1, 0.5, 1.5$ while the other parameters are kept constant as shown in Eq. (19); then, the paths of system (1) are drawn as given in Figs (2a)-(2c).

It is obvious that as the value of r increases with the value of x - species while the value of Z and P decreases towards their minimum values. Further, the species x faces extinction when r decreases, reaching the value $r = 0.1$.

The effect of varying the intra-specific competition parameter on the system's dynamic behaviour was investigate in case of the existence of toxicant (i.e., parameter a). So, system (1) was numerically solved for various values of $a = 0.2, 0.5, 2$ while keeping the other parameters fixed as shown in Eq. (19); the paths of system (1) are drawn in Figs. (3a) - (3c).

Obviously, the increase in the intra-specific competition between the members of species x decreases the value of x decreases while the toxicant concentrations of Z and P increases.

Now, the effect of varying the fraction of the natural depletion rate coefficient of Z (t) that may again enter the environment (i.e., parameter ϑ) on the system's dynamics was analysed. Hence, system (1) was solved for various values

FIGURE 1. Phase plot of the solution of system (1) starting from different initial points.

FIGURE 2. Time series of the solution of system (1) for different values of r ; blue colour for $r = 0.1$; green colour for $r = 0.5$; red colour for $r = 1.5$. (a) The Trajectories of x as a function of time. (b) The Trajectories of Z as a function of time. (c) The Trajectories of P as a function of time.

FIGURE 3. Time series of the solution of system (1) for different values of a ; blue colour for $a = 0.2$; green colour for $a = 0.5$; red colour for $a = 2$. (a) The Trajectories of x as a function of time. (b) The Trajectories of Z as a function of time.(c) The Trajectories of P as a function of time.

FIGURE 4. Time series of the solution of system (1) for different values of v ; blue colour for $v = 0.2$; green colour for $v = 0.6$; red colour for $v = 0.99$. (a) The Trajectories of x as a function of time. (b) The Trajectories of Z as a function of time.(c) The Trajectories of P as a function of time.

of $\vartheta = 0.2, 0.6, 0.99$, while keeping the other parameters constant as shown in Eq. (19). The paths of system (1) are drawn in Figs. (4a)-(4c).

According to these figures, it is observed that as ϑ increases, the value of x decreases while the toxicants concentrations of Z and P increases.

In the following, the effect of varying the depletion of toxicant Z on the system's dynamics was studied by numerically solving system (1) for various values of $d_1 = 0.1, 0.4, 2$ while fixing the values of the other parameters as shown in Eq. (19); the trajectories of system (1) are shown in Figs. (5a)-(5c).

FIGURE 5. Time series of the solution of system (1) for different values of d_1 ; blue colour for $d_1 = 0.1$; green colour for $d_1 = 0.4$; red colour for $d_1 = 2$. (a) The Trajectories of x as a function of time. (b) The Trajectories of Z as a function of time.(c) The Trajectories of P as a function of time.

Obviously, as shown in Fig. (5), when d_1 decreases to the value of $d_1 = 0.1$, the species x faces extinction and the system's solution tends towards the equilibrium point $e_1 = (0, \tilde{Z}, \tilde{P}) = (0, 10.22, 3.4)$. However, as d_1 increases, species x starts increasing while the toxicants concentrations of Z and P are decreased towards the minimum values; hence, system (1) approaches a stable point $e_2 = (\bar{x}, \bar{Z}, \bar{P})$.

The effect of varying the depletion rate of P (i.e. the parameter d_2) on the dynamics of system (1) was also studied. It was observed that system (1) still approaches a positive equilibrium point e_2 in the $\text{Int. } R_+^3$, but with the fact that as the toxicants concentrations decrease due to increase of d_2 , the species x increases.

Now, the effect of the absorption rate of the toxicant Z by species x on the system's dynamics (i.e parameter α) was studied. This requires solving for different values of $\alpha = 0.4, 0.6, 1$ while the other parameters are kept constant as shown in Eq. (19); the paths of system (1) are drawn in Figs. (6a) - (6c).

Obviously, according to Fig. (6), as α increases, the value of x decreases while the values of Z and P increase. Furthermore, when α was approaching the value $\alpha = 1$, the solution of system (1) also approached a globally and asymptotically stable point $e_1 = (0, \tilde{Z}, \tilde{P}) = (0, 1.02, 3.4)$. In this case, Z starts decreasing again while P still increases due to the extinction of x . Consequently, increasing the value of the absorption value α causes the extinction of species x .

However, the system was numerically solved for different values of $k = 0.5, 2, 3$ (the other parameters were fixed as shown in Eq. 19) in order to investigate the effect of the depletion rate coefficient of the pollutants in the system due to the uptake of by x – species on the system's dynamics. The paths of the system are shown in Figs. (7a) -(7c).

According to Fig. (7), as k increases, the value of x – species and the toxicant concentration P also decreases while the toxicant concentration Z keeps increasing. Hence, system (1) still approaches a positive point e_2 .

The effect of the depletion rate coefficient of Z due to their removal by x – species (i.e parameter γ) on the system's dynamics was also numerically investigated by studying for different values of $\gamma = 0.4, 0.9, 2$ while keeping the other parameters constant as shown in Eq. (19). The paths of the system are shown in Figs. (8a) - (8c).

FIGURE 6. Time series of the solution of system (1) for different values of α ; blue colour for $\alpha = 0.4$; green colour for $\alpha = 0.6$; red colour for $\alpha = 1$. (a) The Trajectories of x as a function of time. (b) The Trajectories of Z as a function of time.(c) The Trajectories of P as a function of time.

FIGURE 7. Time series of the solution of system (1) for different values of k ; blue colour for $k = 0.5$; green colour for $k = 2$; red colour for $k = 3$. (a) The Trajectories of x as a function of time. (b) The Trajectories of Z as a function of time.(c) The Trajectories of P as a function of time.

FIGURE 8. Time series of the solution of system (1) for different values of γ ; blue colour for $\gamma = 0.4$; green colour for $\gamma = 0.9$; red colour for $\gamma = 2$. (a)) The Trajectories of x as a function of time. (b) The Trajectories of Z as a function of time.(c) The Trajectories of P as a function of time.

According to the above figures, it is observed that as γ increases, the value of species increases too while the values of Z and P decreases.

Finally, the exogenous input rate coefficient of the pollutant in the environment (i.e parameter Q) was observed to affect the system's dynamics in a similar manner as parameter ϑ .

6. CONCLUSION

A mathematical model for the description of the dynamic behaviour of a system of single species in a PE was proposed and analysed in this article. The system was also discussed in terms of the boundedness, followed by the establishment of the existence conditions for all the possible equilibrium points of the studied system. Analysis was also performed on the dynamic behaviour of the established equilibrium points both locally and globally. Lastly, the analytical results of the system were further evaluated by studying the effect of pollutants on the dynamic behaviour of a single species in the PE. Based on the outcomes of the studies on the impacts of various parameters on the dynamic behaviour of the system, it was concluded that:

1. For appropriate set of parameters values, the dynamical behaviour of system (2.1) approaches to the global asymptotical and stable point, and still co-exists despite the presence of pollutants.
2. In case of decreasing the values of r , the positive equilibrium point $e_2 \in \text{Int} \cdot R_+^2$ become unstable while the paths of system (2.1) tends towards the asymptomatic equilibrium point e_1 in the interior of ZP -space.
3. Furthermore, increases in the value of α destabilizes the positive equilibrium point $e_2 \in \text{Int} \cdot R_+^3$, causing the trajectory of system (2.1) to approach asymptotically to a stable point e_1 in the interior of ZP -space.
4. Finally, increases in the value of d_1 destabilized the positive equilibrium point $e_2 \in \text{Int} \cdot R_+^3$ and then, the trajectory of system (2.1) approaches asymptotically to a stable point e_1 in the interior of ZP -space.

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