2-Visible Submodules and Fully 2-Visible Modules

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ABSTRACT: Let X be a T -module, T is a commutative ring with identity and K be a proper submodule of X . In this paper we introduce the concepts of 2-visible submodules and fully 2-visible modules as a generalizations of visible submodules and fully visible modules resp., where K is said to be 2-visible whenever $K = I^2 K$ for every nonzero ideal I of T and A^T -module X is called fully 2-visible if for any proper submodule of it is 2-visible. Study some of the properties of these concepts also discuss the relationship 2-visible submodules and fully 2-visible modules with 2-pure submodules and other related submodules and modules resp. are given.

Keywords: visible submodules; 2-visible submodules; 2-pure submodules; strongly cancellation module; fully visible modules; fully 2-visible modules

1. INTRODUCTION

In this study, T is a commutative ring with identity and X is unitary module. Anderson and Fuller [1] called the submodule N a pure submodule of X if $IN = N \cap IX$ for every ideal I of T . Ribenboim [2] defined N to be pure in M if $rX \cap N = rN$ for each $r \in T$. Mijbass in 1992 presented the concept of cancellation module "A T -module X is defined to be a cancellation module if $IX = JX$ for ideals I and J of T implies $I = J$ ", [3].

In [4] Mahmood and Buthyna presented the concept of visible submodule" A proper submodule W of a module X over a ring T , so that it achieves $W = IW$ for every nonzero ideal I of T . Many of the properties which characterize this concept have been built add to a lot of important results and features have been submitted in [5]. Also in [6] Mahmood presented the concept of fully visible module, which is defined as the module, in which all submodules are visible.

In this article we made a generalizations of visible submodules to a 2-visible submodules and fully visible modules to fully 2-visible modules, many properties of these concepts have been proved.

Note that: we denote to submodule: subm.

2. 2-VISIBLE SUBMS

Definition 2.1. Let X be a T -module and K be a proper subm. of X . Then K is said to be 2-visible whenever $K = I^2 K$ for every nonzero ideal I of T .

An ideal I of a ring T is called 2-visible if it is 2-visible T -subm. of X .

Remarks and Examples 2.2. 1) A zero subm. of any T -module X is always 2-visible.

2) Consider Z_4 as a Z -module. A subm. $(\bar{2})$ of Z_4 is not 2-visible. Since, if we take the ideal $I = (2)$ then $(\bar{2}) \neq (2)^2(\bar{2}) = (\bar{0})$, more than this for every nonzero ideal A of Z we will get $(\bar{2}) \neq A^2(\bar{2})$.

3) Two subms $(\bar{2})$ and $(\bar{3})$ of Z -module Z_6 are not 2-visible. Note, let $I = (6)$ and $J = (2)$ be two ideals of Z . Then $(\bar{2}) \neq (6)^2(\bar{2}) = (\bar{0})$ also $(\bar{3}) \neq (2)^2(\bar{3}) = (\bar{0})$.

4) Let K be a 2-visible subm. of X . Then for every proper subm. L of X is 2-visible whenever $f : K \rightarrow L$ is an epimorphism.

Proof. Let $f : K \rightarrow L$ be an epimorphism. Then $f(K) = L$, but K is 2-visible subm. of X , therefore for every nonzero ideal A of T we have $K = A^2K$ which implies that $L = f(K) = f(A^2K) = A^2f(K) = A^2L$ and hence L is 2-visible.

5) Let X_1 and X_2 be two T -modules and $\psi : X_1 \rightarrow X_2$ be an T -homomorphism. If K is 2-visible subm. X_2 of, then $\psi^{-1}(K)$ is also 2-visible subm. of X_1 .

Proof. Since K is 2-visible subm. of X_2 , then for every nonzero ideal I of T , we have $I^2K = K$. Therefore $I^2\psi^{-1}(K) = \psi^{-1}(I^2K) = \psi^{-1}(K)$.

Proposition 2.3. Let X be a T -module and E be a 2-visible subm. of X . If L is a subm. of E , then E/L is 2-visible subm. of X/L .

Proof. Let $(0) \neq A$ be an ideal of T . Now, $A^2(E/L) = (A^2E + L)/L$. But $A^2E = E$ (since E is 2-visible subm. of X).

Then $(A^2E + L)/L = (E + L)/L = E/L$.

Therefore E/L is 2-visible subm. of X/L .

Proposition 2.4. Let X be a T -module and D, L be two subms of X . If D, L are 2-visible subms, then $D + L$ is 2-visible subm.

Proof. Let A be a nonzero ideal of T . Then $A^2(D + L) = A^2D + A^2L = D + L$ (since D and L are 2-visible subms). Therefore $D + L$ is 2-visible subm. of X .

Remark 2.5. As a generalization of Proposition (2.4), we get: if $\{N_k\}_{k=1}^n$ is a finite collection of subms of a T -module X and N_k is 2-visible subm. for all k , then the sum of all these subms is 2-visible subm. of X .

Proposition 2.6. Let X be a T -module and N be a 2-visible subm. of X . Then every subm. of N is also 2-visible.

Proof. Let N be a 2-visible subm. of a T -module X and let K be a proper subm. of N , that is $K \subseteq N$. Since $N = I^2N$ for every nonzero ideal I of T by hypothesis. Then $K \subseteq I^2N$ which implies that $I^2N + K = I^2N \dots (1)$

Also, from the above inclusion, we get $I^2K \subseteq I^2N$.

And hence $I^2N + I^2K = I^2N \dots (2)$.

Form (1) and (2), we get $I^2N + K = I^2N + I^2K$ and hence $K = I^2K$. Therefore K is 2-visible subm.

Remark 2.7. The converse of Proposition (2.6) need not to be correct for example:

The module Z_{12} as Z_{36} -module. Since $(\bar{0})$ is contains in any subm. of any T -module X and $(\bar{0})$ is 2-visible subm. by Remarks and Examples(2.2). But a subm. $(\bar{6})$ of module Z_{12} is not 2-visible, since there is a nonzero ideal of Z_{36} that is (2) such that $(\bar{6}) \neq (2)^2(\bar{6}) = (\bar{0})$. Therefore $(\bar{6})$ is not 2-visible subm. in Z_{12} .

Corollary 2.8. Let N_1, N_2 , be two subms of a T -module X . If either N_1 or N_2 is 2-visible subm. of X , then $N_1 \cap N_2$ is also 2-visible subm.

Proof. It is clearly that $N_1 \cap N_2 \subseteq N_1$ and $N_1 \cap N_2 \subseteq N_2$ but N_1 is 2-visible, then by proposition (2.6), $N_1 \cap N_2$ is also 2-visible. Similarly with N_2 is 2-visible.

As a directly result of Corollary (2.8), we give the following generalization.

Corollary 2.9. Let $\{N_i\}_{i=1}^n$ be a family of subms of a T -module X such that at least one of them is 2-visible, then $\cap_{i=1}^n N_i$ is 2-visible subm.

Proof. We know that $\cap_{i=1}^n N_i \subseteq N_i \forall i$. Then the result is directly from Proposition (2.6).

Proposition 2.10. For each nonzero ideal A of T and for each nonempty collection $\{W_\alpha\}$ of 2-visible subm. of an T -module X . We have $A^2(\cap_\alpha W_\alpha) = \cap_\alpha A^2W_\alpha$.

Proof. It is known that $\cap_\alpha W_\alpha \subseteq W_\alpha$ for each α , but W_α is 2-visible subm.odule for each α and hence $A^2W_\alpha = W_\alpha$ for each α , also by Proposition (2.6), we get $\cap_\alpha W_\alpha$ is 2-visible subm. of X .

Now, $\cap_\alpha A^2W_\alpha = \cap_\alpha W_\alpha = A^2(\cap_\alpha W_\alpha)$, (since W_α is 2-visible subm. for each α).

Definition 2.11. [7]

Let X be a T -module. A subm. N of X is called 2-pure subm. of X if for each ideal I of T , $I^2X \cap N = I^2N$.

Proposition 2.12. Let N be a 2-visible subm. of a T -module X . Then N is 2-pure subm. of X .

Proof. Since N is 2-visible subm. of X , then $N = I^2N$ for every nonzero ideal I of T . $I^2N \subseteq I^2X$ and $I^2N \subseteq N$.

Therefore $I^2N \subseteq I^2X \cap N$(1)

Also $I^2X \cap N \subseteq N = I^2N$(2)(since N is 2-visible subm.).

Therefore from (1) and (2) we get $I^2X \cap N = I^2N$. Which complete this proof.

Remark 2.13. The converse of Proposition (2.12) since if we take Z_4 as a Z -module and $N = \{\bar{0}, \bar{2}\}$ is a subm. of Z_4 . Then for every ideal $I = nZ$ of Z where n is any positive integer, we have $I^2N = I^2Z_4 \cap N$, that is N is 2-pure subm. of Z_4 but it is not 2-Visible by Remarks and Examples (2.2).

Proposition 2.14. Let X be a module on T , N be a 2-visible subm. of X . If A is a subm. of X such that $N \subseteq A$, then N is a 2-pure subm. of A .

Proof. Let P be an ideal of T and $P \neq (0)$. Since N is 2-visible subm. in X , then $N = P^2N$ hence by Proposition (2.12) we get N is 2-pure subm. then we obtain since $N \subseteq A \subseteq X$, so we get :

$P^2N = P^2A \cap P^2N$ (since $N \subseteq A$, implies $P^2N \subseteq P^2A$).

$P^2N = P^2A \cap (N \cap P^2X) = (P^2A \cap P^2X) \cap N = P^2A \cap N$

Therefore N is 2-pure in A .

Proposition 2.15. Let X be a T -module and N be a 2-pure subm. in X . If A is a 2-visible subm. of N , then A is 2-pure in X .

Proof. For every nonzero ideal P of T , we have $A = P^2A$ (since A is 2-visible subm.) and N is 2-pure in X , then $N \cap P^2X = P^2N$.

But $A \subseteq N$. Therefore $P^2A \cap P^2N = P^2A \cap N \cap P^2X$. Note $P^2A \subseteq P^2N$, and A is visible subm. of N then $P^2A = A \cap N \cap P^2X = A \cap P^2X$. Therefore A is 2-pure in X .

Definition 2.16. 1) A T -module X is called strongly cancellation module if for each ideals I and J of T such that $IN = JN$ then $I = J$ for every subm.odule N of X [5].

2) Let K, W be two T -subms of X . Then the residual of K by W is the set of all $s \in T$ such that $sW \subseteq K$ and denoted by $(K : W)$ The annihilator of X is written as $(0 : X)$ and denoted by $\text{ann}_T(X)$, if $\text{ann}_T(X)$ is equal to zero, then X is said to be faithful [8].

3) A T -module X is called cyclic if and only if there exists $\text{ann}_T(X)$ such that $X = Tx$ [9].

Proposition 2.17. Let D be a 2-visible subm. of a strongly cancellation T -module X . Then $\text{ann}(I^2D) = \text{ann}(I^2)$, for every nonzero ideal I of T .

Proof. Let $x \in \text{ann}_X(I^2)$ Then $xI^2 = 0$ and hence $xI^2D = 0$ which implies that $x \in \text{ann}(I^2D)$ Therefore $\text{ann}(I^2) \subseteq \text{ann}(I^2D)$

Now let $y \in \text{ann}(I^2D)$ Then $yI^2D = 0$, but D is 2-visible subm., then $yD = 0$ and hence $yD = 0D$, we have X is strongly cancellation module. Then $y = 0$. Thus $yI^2 = 0$ and hence $y \in \text{ann}(I^2)$. Finally We obtain $\text{ann}(I^2D) \subseteq \text{ann}(I^2)$.

Therefore $\text{ann}(I^2D) = \text{ann}(I^2)$.

3. FULLY 2-VISIBLE MODULES

Definition 3.1. A T -module X is called fully 2-visible if for any proper subm. of it is 2-visible. A ring T is named fully 2-visible if it is fully 2-visible T -module.

Remarks and Examples 3.2. 1) If P is a prime number, then Z_p as a Z_{p^n} -module, $n > 1$ is a fully 2-visible.

2) The Z -module Z_p always is fully 2-visible, P is prime number.

3) Consider the module Z over Z is not fully 2-visible since, for every proper subm. nZ ,

n is positive integer, there is no a nonzero ideal J of T such that $nZ = J^2(nZ)$, for example if we take the subm. $2Z$ of a module Z , then $2Z \neq (3Z)^2(2Z)$ where $3Z$ is an ideal of Z . Therefore $2Z$ is not 2-visible.

- 4) The module $Z_p \oplus Z$ over a ring Z is not fully 2-visible, P is prime number, since $\frac{(\bar{0})}{(\bar{0}) \oplus Z}$ be a proper subm. of $Z_p \oplus Z$. Then $(\bar{0}) \oplus Z \neq (P)^2((\bar{0}) \oplus Z) = (\bar{0}) \oplus P^2Z$. This means that we found an ideal in T does not achieve definition.
- 5) Z_{12} as Z_{36} -module is not fully 2-visible, every proper subm $\frac{(\bar{0})}{2}, \frac{(\bar{0})}{4}, \frac{(\bar{0})}{8}$ there is no a nonzero ideal I of Z_{36} such that $I = IN$.
- 6) Let X be an T -module. Then every subm. of fully 2-visible module is also fully 2-visible.

Proof. Suppose that X is fully 2-visible module and let N be a proper subm. of X . To prove that N is also fully 2-visible. Now let K be a proper subm. of N . Hence $K \subset X$ and by hypothesis on X , we get K is 2-visible subm., therefore N is fully 2-visible.

Proposition 3.3. A T -module X is fully 2-visible iff X/K is fully 2-visible T -module for every proper subm. K of X .

Proof. Suppose that X is fully 2-visible and W/K be a proper subm. of X/K and let I be a nonzero ideal of T . To prove W/K is 2-visible. $I^2(W/K) = (I^2W + K)/K$, but X is fully 2-visible module then W is 2-visible subm. of X (that is $I^2W = W$). Therefore $I^2(W/K) = (I^2W + K)/K = W/K$ and hence X/K is fully 2-visible. The opposite side is clear by taking $K = 0$.

Corollary 3.4. Let X, X' be two T -modules and $f : X \rightarrow X'$ is an T -epimorphism. If X is a fully 2-visible module, then X' is a fully 2-visible module.

Proof. We have $f : X \rightarrow X'$ is an epimorphism and X is a fully 2-visible module. Then by the first isomorphism theorem, we get $X/\ker f \cong X'$. But $X/\ker f$ is a fully 2-visible module by Proposition 3.3. Therefore X' is a fully visible module.

Proposition 3.5. Let $X = L \oplus N$ be an T -module, where L and N are two subms of X such that $\text{ann}L + \text{ann}N = T$. Then L and N are fully 2-visible modules iff X is a fully 2-visible module.

Proof. Let P be a proper subm. of X . Then $P = L_1 \oplus N_1$ where L_1 and N_1 are subms. of L and N respectively, also let $0 \neq I$ be an ideal of T , then $I^2P = I^2(L_1 \oplus N_1) = I^2L_1 \oplus I^2N_1 = L \oplus N = P$ (since L_1 and N_1 are subms of L and N each of L and N is fully 2-visible. Therefore X is fully visible module.

The opposite side: since $L \subseteq X = L \oplus N$ and $N \subseteq X = L \oplus N$ but X is fully 2-visible module, then by Remarks and Examples (3.1) part(6), we obtain that L and N are fully 2-visible modules.

Proposition 3.6. Let X_1 and X_2 be two T -modules such that $\text{ann}(X_1) + \text{ann}(X_2) = T$, then $X = X_1 \oplus X_2$ is fully 2-visible module iff X_1 and X_2 are fully 2-visible module.

Proof. Suppose that $X = X_1 \oplus X_2$ is fully 2-visible module on T . Let $\psi_i : X \rightarrow X_i$ be the natural projection map, $\forall i = 1, 2$. Note ψ_i is an epimorphism and X is fully 2-visible module then by Corollary 3.4, we get X_i is fully 2-visible module.

Proposition 3.7. Let X_1, X_2 , be two T -modules and W_1, W_2 , are two subms. of X_1, X_2 , resp. such that $\text{ann}_T X_1 + \text{ann}_T X_2 = T$. Then W_1, W_2 , are fully 2-visible T -subm. iff $W_1 \oplus W_2$ is fully 2-visible T -subm. of $X_1 \oplus X_2$.

Proof. Let $K_1 \oplus K_2$ be a proper subm. of $W_1 \oplus W_2$ where K_1 and K_2 are subms of W_1, W_2 resp. and for each a nonzero ideal I of T . To prove $K_1 \oplus K_2 = I^2(K_1 \oplus K_2)$.

Now $I^2(K_1 \oplus K_2) = I^2K_1 \oplus I^2K_2$, but $K_1 \subseteq W_1$ and W_1 is fully 2-visible module, then K_1 is 2-visible subm. and $I^2K_1 = K_1$, similarly $I^2K_2 = K_2$.

Therefore $I^2(K_1 \oplus K_2) = K_1 \oplus K_2$ and hence $W_1 \oplus W_2$ is fully 2-visible T -module.

The opposite side: let K_1 be a proper subm. of W_1 . Then $K_1 \subseteq W_1 \subseteq W_1 \oplus W_2$, but $W_1 \oplus W_2$ is fully 2-visible module, then K_1 is 2-visible and hence W_1 is fully 2-visible module. Similarly, we have W_2 is fully visible module.

It is clear that every direct summand is a subm. of X , then by Remarks and Examples (3.1) part(6), we obtain the following known fact.

Remark 3.8. A direct summand of fully 2-visible T -module X is also fully 2-visible.

Remark 3.9. If the direct summand of X is fully 2-visible, then it is not necessary that the module X is fully 2-visible and the following example shows:

The Z module $X = Z \oplus Z_3$ is not fully 2-visible Z -module, since $Z \oplus (\bar{0})$ is a proper subm. of X , but there is no a nonzero ideal I of Z such that $(z \oplus (\bar{0})) = I^2(z \oplus (\bar{0})) = I^2Z$, while Z_3 as a Z -module is fully 2-visible module by Remarks and Examples (3.1).

Remark 3.10. *It is not necessary that $X^2 = X \oplus X$ is fully 2-visible T -module if X is fully 2-visible T -module.*

It is clear that Z_3 as Z -module is fully 2-visible by Remarks and Examples(3.1). But $X = Z_3 \oplus Z_3$ as Z -module is not 2-visible, since if we take $N = (\bar{0}) \oplus Z_a$ be a subm. of X . Then N is not 2-visible, since if $I = (2)$ is an ideal of Z , we get $N \neq (2)^2N$.

Proposition 3.11. *Let X be a module on T . Then the following are equivalent.*

- (1) X is fully 2-visible module.
- (2) Every cyclic subm. of X is a 2-visible subm. of X .

Proof. (1) $\Rightarrow$ (2), It follows by Definition (3.1).

(2) $\Rightarrow$ (1) Suppose that every cyclic subm. of X is 2-visible. Now, let N be a proper subm. of X and I be a nonzero ideal of T . Take $x \in N$, then $(x) \subseteq N$, implies (x) is 2-visible subm. of X .

Therefore $x \in (x) = I^2(x) \subseteq I^2N$ and hence $x \in I^2N$. We get from this $N \subseteq I^2N$, also we have $I^2N \subseteq N$. This achieves a definition of 2-visible subm. Which mean $N = I^2N$. Therefore X is fully 2-visible.

4. CONCLUSIONS

In this paper we study the concept of 2-visible subm., we get the result: A zero subm. of any T -module X is always 2-visible and the inverse image of 2-visible subm. is 2-visible subm., also the sum of two 2-visible subms is 2-visible subm. If N be a 2-visible subm. of X , then every subm. of N is also 2-visible subm., but the converse incorrect see Remark(2.7). Let N_1, N_2 be two subms of a T -module X . If either N_1 or N_2 is 2-visible subm. of X , then $N_1 \cap N_2$ is also 2-visible subm. The relation between 2-visible subm. and 2- pure subm. where every 2-visible subm. is 2- pure subm, but the converse incorrect see Remark(2.13).

Next, we study the concept of fully 2-visible, we get The Z -module Z_p always is fully 2-visible, P is prime number, but The module $Z_p \oplus Z$ over a ring Z is not fully 2-visible and every subm. of fully 2-visible module is also fully 2-visible. If $X = L \oplus N$ be an T -module, where L and N are two subms of X such that $\text{ann}L + \text{ann}N = T$. Then L and N are fully 2-visible modules iff X is a fully 2-visible module. Finally, A direct summand of fully 2-visible T -module X is also fully 2-visible, but the direct summand of X is fully 2-visible, then it is not necessary that the module X is fully 2-visible see Remark (3.9).

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