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Computation of the Stabilizing Solution to Nash-Riccati Equations

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ABSTRACT: Consider a new iterative scheme of linearized Newton's method to calculate the minimal nonnegative solution of a nonsymmetric Riccati equation associated with a game model. The minimal solution is important to find Nash strategies in a game for positive systems. The Newton procedure is applied to work out a nonnegative solution of this type of equations. Our proposal is effective one because it employs small number of matrix multiplication at each iteration step and there is a variant to exploit the block structure of matrix coefficients of the Nash-Riccati equation. Moreover, in this reason, it is easy to extend the proposed iterative modification depending on the number of players of a given game model. We provide a numerical example where compare the results from experiment with the proposed iteration of linearized Newton's method.

Keywords: Game Model; Nash-Riccati equation; Nash Equilibrium; Stabilizing Solution.
AMS Subject Classification: 15A24, 65F45.

1. INTRODUCTION

We ponder the nonsymmetric Riccati equation with matrix coefficients in the form

$$\mathcal{R}(\mathcal{X}) = -D\mathcal{X} - \mathcal{X}A - Q + \mathcal{X}S\mathcal{X}, \qquad \mathcal{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \tag{1}$$

where $(-A)$ is a real $n \times n$ coefficient, and D is a diagonal 2×2 block matrix with entries A^T. The matrices S and Q are block matrices of the form $S = (S_1 \;\; S_2)$, $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$.

In addition, we know $S_j = B_j R_{jj}^{-1} B_j^T$ $(S_j = S_j^T)$ is a nonpositive matrix, B_j is an $n \times m_j$ nonnegative matrix, Q_j is an $n \times n$ symmetric nonnegative matrix, R_{jj} is an $m_j \times m_j$ matrix with negative entries for $j = 1, 2$ and X_1, X_2 are $n \times n$ unknown matrices.

The definition of the stabilizing solution $\tilde{\mathcal{X}} = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ of (1) is defined [1] as a left-right stabilizing solution. It uses to construct both matrices $A - S_1\tilde{X}_1 - S_2\tilde{X}_2$ and $\begin{pmatrix} A^T - \tilde{X}_1 S_1 & -\tilde{X}_1 S_2 \\ -S_1\tilde{X}_2 & A^T - \tilde{X}_2 S_2 \end{pmatrix}$ which are stable. The Newton method to determine $\tilde{\mathcal{X}}$ is investigated and a convergence proof is derived under several assumptions [1]. The Nash equilibria theory applied to a game model on positive systems is presented in [2]. The theory is founded on the concept of deterministic feedback behaviour and the open loop approach. The linearized Newton method (LNM) to determine $\tilde{\mathcal{X}}$ is introduced in [5].Moreover, the LNM is numerically investigated by Baeva [4].

Guo [5] has been proved that if the matrix K = [-A S; -Q -D] is an M-matrix then equation (1) has minimal nonnegative solution. Further on, Guo [6] has been derived the property of $\mathcal{R}(\mathcal{X}) = 0$ that if K is a regular M-matrix and all diagonal

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entries of $(-A)$ are positive then there is the nonnegative solution Φ with $-A+S\Phi$ is a regular M-Matrix. Therefore, $\Phi < X$ for any nonnegative solution of the inequality $\mathcal{R}(\mathcal{X}) \leq 0$, which means that Φ is the minimal nonnegative one.

In this paper, we consider an improvement of the linearized Newton method to determine the stabilizing solution of (1) which has nonnegative entries. In fact, that is the minimal nonnegative solution. We derive a convergence proof for this modification. In addition, we display a decoupled variant of this modification.

We propose a modification of the linearized Newton method to determine the stabilizing solution. The proposed iterative scheme is more productive than the LNM. Our computer realization requires a smaller number of matrix multiplications. The algorithm uses less CPU time to compute the stabilizing solution than LNM. Moreover, the algorithm of the introduced iteration can be reorganized to obtain a parallel realization. Finally, we done numerical experiments to certify the advantages of the introduced scheme and compare two solvers.

We apply the some facts and properties of matrix algebra and specially of branch of nonnegative matrices. We denote the set of real $s \times q$ matrices with $\mathbf{R}^{s\times q}$. Matrices are denoted in the following way $A = (a_{ij}), 1 \leq i \leq m,\ 1 \leq j \leq n$, in shortly $A = (a_{ij}) \in \mathbf{R}^{m\times n}$. The nonnegative matrices are defined in an elementwise order relation. Each nonnegative matrix has nonnegative entries, i.e. A is nonnegative if $a_{ij} \geq 0$ for all indexes i, j. In addition, the inequality $F \geq R (F > R)$ for $F = (f_{ij}), R = (r_{ij})$ means that $f_{ij} \geq r_{ij} (f_{ij} > r_{ij})$ for all indexes i and j.

A square matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is called a Z-matrix if it has nonpositive off-diagonal entries. Moreover, any Z-matrix A can be presented as $A = \alpha I_n - N$, where I_n is the identity matrix and N is a nonnegative one. We need to introduce M-matrices as a tool in our investigation. If $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N, the matrix A is an M-matrix. Each M-matrix is a Z-matrix with if $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N. More specially, if $\alpha > \rho(N)$ it is a nonsingular M-matrix and if $\alpha = \rho(N)$ it is a singular one. In addition, any matrix is a stable one, if its eigenvalues has negative real parts.

2. ITERATIVE METHODS AND CONVERGENCE ANALYSIS

The linearized Newton method (LNM) was introduced by C. Ma and H. Lu in [5]. Set $Z_0 = 0$, we compute matrix sequences $\{Y_i\} = \begin{pmatrix} Y_1^{(i)} \\ Y_2^{(i)} \end{pmatrix}\}$ and $\{Z_i = \begin{pmatrix} Z_1^{(i)} \\ Z_2^{(i)} \end{pmatrix}\}$ via following iterations

$$Y_{i+1}(\gamma I_n + A - S Z_i) = (\gamma I_{2n} - D) Z_i - Q \tag{2}$$

$$(\gamma I_{2n} + D - Y_{i+1} S) Z_{i+1} = Y_{i+1}(\gamma I_n - A) - Q, \tag{3}$$

for $i = 0, 1, 2, ...$ and $\gamma < 0$, as sequence $\{Z_i\}$ converge to the solution $\tilde{K}$, when i converge to the infinity [5].The convergence properties are proved in [5].

Here, we slightly modify iteration (2)-(3) to upgrade linearized Newton's method:

$$\mathcal{Y}^{(k)}(\gamma I_n + A) = (\gamma I_{2n} - D + \mathcal{X}^{(k)} S)\,\mathcal{X}^{(k)} - Q \tag{4}$$

$$(\gamma I_{2n} + D)\mathcal{X}^{(k+1)} = \mathcal{Y}^{(k)}(\gamma I_n - A + S\,\mathcal{Y}^{(k)}) - Q \tag{5}$$

We introduce a few properties of matrix sequences defined by (4)-(5)

Lemma 1. *We construct two sequences* $\{\mathcal{X}^{(k)}, \mathcal{Y}^{(k)}\}_{k=0}^{\infty}$ *applying iteration (4) - (5) for an initial value* $\mathcal{X}^{(0)} = 0$. *The following matrix identities are satisfied for* $k = 0, 1, \ldots, \infty$*:*

$$\begin{aligned}\text{(i)}\ \ (\mathcal{Y}^{(k)} - \mathcal{X}^{(k)})(\gamma I + A) &= (\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)})(\gamma I - A) \\ &+ \mathcal{X}^{(k)} S (\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)}) + (\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)}) S\, \mathcal{Y}^{(k-1)},\end{aligned}$$

$$\begin{aligned}\text{(ii)}\ (\gamma I_{2n} + D)(\mathcal{X}^{(k+1)} - \mathcal{Y}^{(k)}) &= (\gamma I_{2n} - D)(\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}) \\ &+ (\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}) S\, \mathcal{Y}^{(k)} + \mathcal{X}^{(k)} S (\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}).\end{aligned}$$

Moreover, if $\tilde{\mathcal{X}}$ *is an exact solution of* $\mathcal{R}(\mathcal{X}) = 0$ *the identities can be verified:*

$$\begin{aligned}\text{(iii)}\ \ (\mathcal{Y}^{(k)} - \tilde{\mathcal{X}})(\gamma I + A) &= (\gamma I_{2n} - D)(\mathcal{X}^{(k)} - \tilde{\mathcal{X}}) \\ &+ (\mathcal{X}^{(k)} - \tilde{\mathcal{X}}) S\, \mathcal{X}^{(k)} + \tilde{\mathcal{X}} S (\mathcal{X}^{(k)} - \tilde{\mathcal{X}}).\\ \text{(iv)}\ (\gamma I_{2n} + D)(\mathcal{X}^{(k+1)} - \tilde{\mathcal{X}}) &= (\mathcal{Y}^{(k)} - \tilde{\mathcal{X}})(\gamma I - A) \\ &+ (\mathcal{Y}^{(k)} - \tilde{\mathcal{X}}) S\, \mathcal{Y}^{(k)} + \tilde{\mathcal{X}} S (\mathcal{Y}^{(k)} - \tilde{\mathcal{X}}).\end{aligned}$$

Proof. The proof is completed by direct calculations and matrix manipulations. We rewrite equation (5) for $\mathcal{X}^{(k)}$ and consider the difference $\mathcal{Y}^{(k)}(\gamma I + A) - (\gamma I_{2n} + D)\mathcal{X}^{(k)}$. After some matrix calculations we obtain the matrix identity (i). Subtracting (4) from (5) we derive (ii).

We apply matrix identities derived in Lemma 1 to obtain some convergence properties of introduced scheme (4)-(5). The main contribution in the paper is the convergence analysis for (4)-(5). It is attained in the following theorem.

Theorem 1. *Assume matrices $A, D, S = (S_1, S_2)$, and $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ are coefficients of matrix equation $\mathcal{R}(\mathcal{X}) = 0$. Suppose, there exists a negative number $\gamma < 0$, which guarantees that $-(\gamma I_n + A)$ is an M-matrix.*

The sequences $\{\mathcal{X}^{(k)}, \mathcal{Y}^{(k)}\}_{k=0}^{\infty}$ where matrices $\mathcal{X}^{(k)}, \mathcal{Y}^{(k)}$ are computed via (4) - (5) satisfy inequalities:

$\tilde{\mathcal{X}} \geq \mathcal{X}^{(k+1)} \geq \mathcal{Y}^{(k)} \geq \mathcal{X}^{(k)}$ for $k = 0, 1, \ldots$, for an exact solution $\tilde{\mathcal{X}}$ of $\mathcal{R}(\mathcal{X}) = 0$.

Moreover, the convergence property is fulfilled:

The matrix sequence $\{\mathcal{X}^{(k)}\}_{k=0}^{\infty}$ converges to the minimal solution $\tilde{\mathcal{X}}$ to Riccati equation $\mathcal{R}(\mathcal{X}) = 0$. The solution is nonnegative and stable.

Proof. We begin with the facts that $(\gamma I_n + A)^{-1} \leq 0$ and $(\gamma I_n + A^T)^{-1} \leq 0$. The matrix coefficients are $Q \geq 0$ and $S \leq 0$. Matrices $\gamma I_n - A$, and $\gamma I_{2n} - D$ are nonpositive. We create sequences $\{\mathcal{X}^{(k)}, \mathcal{Y}^{(k)}\}_{k=0}^{\infty}$ applying recursive equations (4) - (5) with $\mathcal{X}^{(0)} = 0$ and $\gamma < 0$.

For $k = 0$ we obtain $\mathcal{Y}^{(0)}(\gamma I_n + A) = -Q \leq$ and thus $Y_i^{(0)} = -Q_i(\gamma I_n + A^T)^{-1} \geq 0$. And $\mathcal{Y}^{(0)} \geq \mathcal{X}^{(0)} = 0$ and $\mathcal{Y}^{(0)} - \mathcal{X}^{(0)} \geq 0$.

Assume the inequalities $\mathcal{X}^{(k)} \geq \mathcal{Y}^{(k-1)} \geq \mathcal{X}^{(k-1)} \geq 0$ true for some integer k. It is true that $\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)} \geq 0$, and $\mathcal{Y}^{(k-1)} - \mathcal{X}^{(k-1)} \geq 0$.

We shall prove that $\mathcal{X}^{(k+1)} \geq \mathcal{Y}^{(k)} \geq \mathcal{X}^{(k)} \geq 0$.

Applying Lemma 1 (i), we get

$$(\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}) = W^{(k)} \, (\gamma I_n + A)^{-1},$$

where

$$W^{(k)} := (\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)})(\gamma I - A) + \mathcal{X}^{(k)} S (\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)}) + (\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)}) S \mathcal{Y}^{(k-1)} \leq 0,$$

because $S \leq 0$ and $\mathcal{X}^{(k)} S (\mathcal{X}^{(k)} - \mathcal{Y}^{(k-1)}) \leq 0$. Thus $(\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}) \geq 0$.

Further on, according to Lemma 1 (ii) we have

$$(\mathcal{X}^{(k+1)} - \mathcal{Y}^{(k)}) = (\gamma I_{2n} + D)^{-1} \, H^{(k)},$$

where

$$H^{(k)} := (\gamma I_{2n} - D)(\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}) + (\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}) S \mathcal{Y}^{(k)} + \mathcal{X}^{(k)} S (\mathcal{Y}^{(k)} - \mathcal{X}^{(k)}) \leq 0.$$

Thus $(\mathcal{X}^{(k+1)} - \mathcal{Y}^{(k)}) \geq 0$.

We conclude the matrix sequences $\{\mathcal{X}^{(k)}, \mathcal{Y}^{(k)}\}_{k=0}^{\infty}$ are monotone increasing. We have to prove that they are bonded above. Consider any exact nonnegative solution $\tilde{\mathcal{X}}$ of $\mathcal{R}(\mathcal{X}) = 0$. We shall prove that the solution is an upper bound of the matrix sequences.

For $k = 0$, we have $\tilde{\mathcal{X}} \geq \mathcal{X}^{(0)} = 0$, and (Lemma 1 (iii))

$$\mathcal{Y}^{(0)} - \tilde{\mathcal{X}} = [\,-(\gamma I_{2n} - D)\tilde{\mathcal{X}} - \tilde{\mathcal{X}} S \tilde{\mathcal{X}}\,](\gamma I + A)^{-1} \; \leq 0.$$

Using the equality for $k = 0$:

$$(\gamma I_{2n} + D)(\mathcal{X}^{(1)} - \tilde{\mathcal{X}}) = (\mathcal{Y}^{(0)} - \tilde{\mathcal{X}})(\gamma I - A) + (\mathcal{Y}^{(0)} - \tilde{\mathcal{X}}) S \mathcal{Y}^{(0)} + \tilde{\mathcal{X}} S (\mathcal{Y}^{(0)} - \tilde{\mathcal{X}}) \geq 0,$$

we conclude $(\mathcal{X}^{(1)} - \tilde{\mathcal{X}}) \leq 0$.

Thus

$$\tilde{\mathcal{X}} \geq \mathcal{X}^{(1)} \geq \mathcal{Y}^{(0)} \geq \mathcal{X}^{(0)}.$$

Assume

$$\tilde{\mathcal{X}} \geq \mathcal{X}^{(k+1)} \geq \mathcal{Y}^{(k)} \geq \mathcal{X}^{(k)}.$$

We shall prove
$$\tilde{\mathcal{X}} \geq \mathcal{X}^{(k+2)} \geq \mathcal{Y}^{(k+1)} \geq \mathcal{X}^{(k+1)}.$$

Applying the identity Lemma 1 (iii) for $k+1$, we obtain
$$(\mathcal{Y}^{(k+1)} - \tilde{\mathcal{X}})(\gamma I + A) = (\gamma I_{2n} - D)(\mathcal{X}^{(k+1)} - \tilde{\mathcal{X}}) + (\mathcal{X}^{(k+1)} - \tilde{\mathcal{X}})S\mathcal{X}^{(k+1)} + \tilde{\mathcal{X}}S(\mathcal{X}^{(k+1)} - \tilde{\mathcal{X}})$$

The conclusion is $(\mathcal{Y}^{(k+1)} - \tilde{\mathcal{X}}) \leq 0$ because the right-hand is nonnegative and $(\gamma I + A)^{-1} \leq 0$. Analogously, based on Lemma 1 (iv) for $k+1$ we find the inequality $(\mathcal{X}^{(k+2)} - \tilde{\mathcal{X}}) \leq 0$.

Thus
$$\tilde{\mathcal{X}} \geq \mathcal{X}^{(k+2)} \geq \mathcal{Y}^{(k+1)} \geq \mathcal{X}^{(k+1)}.$$

Therefore, both matrix sequences converge to the same matrix $\hat{\mathcal{X}}$. Letting $k \to \infty$ in (4) - (5) one concludes that $\hat{\mathcal{X}}$ is a solution of $\mathcal{R}(\mathcal{X}) = 0$.

Assume there is another solution $\mathcal{Z}$ with $\mathcal{Z} \leq \hat{\mathcal{X}}$. There exists a large index s such that $X^{(s+1)} \geq \mathcal{Z} \geq \mathcal{Y}^{(s)} \geq \mathcal{X}^{(s)}$.

Applying Lemma 1 (iv) (k=s) for $\tilde{\mathcal{X}} = \mathcal{Z}$, we get
$$\begin{aligned}(\gamma I_{2n} + D)(\mathcal{X}^{(s+1)} - \mathcal{Z}) &= (\mathcal{Y}^{(s)} - \mathcal{Z})(\gamma I - A) \\ &\quad + (\mathcal{Y}^{(s)} - \mathcal{Z})S\mathcal{Y}^{(s)} + \tilde{\mathcal{X}}S(\mathcal{Y}^{(s)} - \mathcal{Z}).\end{aligned}$$

We rewrite
$$(\gamma I_{2n} + D)(\mathcal{X}^{(s+1)} - \mathcal{Z}) = V^{(s)},$$
with $V^{(s)} \geq 0$. Then $(\mathcal{X}^{(s+1)} - \mathcal{Z}) = (\gamma I_{2n} + D)^{-1} V^{(s)} \leq 0$. The fact "$X^{(s+1)} - \mathcal{Z}$ is nonpositive" is a contradiction with the assumption $X^{(s+1)} \geq \mathcal{Z}$. Therefore the solution $\hat{\mathcal{X}}$ is the minimal one, i.e. it is the stabilizing solution.

3. COMPUTATIONAL BEHAVIOUR

To determine the minimal nonnegative solution of $\mathcal{R}(\mathcal{X}) = 0$ the introduced iteration (4)-(5) is decoupled in the form:
$$Y_1^{(k)}(\gamma I_n + A) = (\gamma I_n - A^T + X_1^{(k)} S_1 + X_2^{(k)} S_2) X_1^{(k)} - Q_1 \tag{6}$$
$$Y_2^{(k)}(\gamma I_n + A) = (\gamma I_n - A^T + X_1^{(k)} S_1 + X_2^{(k)} S_2) X_2^{(k)} - Q_2 \tag{7}$$
$$(\gamma I_n + A')X_1^{(k+1)} = Y_1^{(k)}(\gamma I_n - A + S_1 Y_1^{(k)} + S_2 Y_2^{(k)}) - Q_1 \tag{8}$$
$$(\gamma I_n + A')X_2^{(k+1)} = Y_2^{(k)}(\gamma I_n - A + S_1 Y_1^{(k)} + S_2 Y_2^{(k)}) - Q_2. \tag{9}$$

We set initial values $X_1^{(0)} = X_2^{(0)} = 0 \in \mathbf{R}^{n \times n}$.

In our experiment we replace iteration (4)-(5) with decoupled variant (6)-(9). Actually, the decoupled modification of linearized Newton's method, named DMLNM, was introduced by Tanov [7]. However, in his paper a convergence proof for DMLNM is not given. In this paper we fill that gap and the convergence analysis is presented in Theorem 1.

We present a numerical example where two mentioned iterations LNM (2)-(3) and DMLNM (6)-(9) are enforced to find the minimal nonnegative solution of the corresponding matrix equation. The computed solution is employed to create Nash equilibrium strategies in a two player game model. We compare results obtained from above iterations. The matrix coefficients Q_1, Q_2, R_{11}, and R_{22} for $n = 4$ are created under the Matlab depiction.

Example. We define the coefficients of (1) as follows:
$$A = \begin{pmatrix} -2.74 & 0.06 & 0.015 & 0.099 \\ 0.2 & -2.5 & 0.064 & 0.08 \\ 0.004 & 0.15 & -2.56 & 0.09 \\ 0.14 & 0.12 & 0.21 & -2.57 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.5938 \\ 0.2985 \\ 0.49 \\ 0.98 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 2.8 & 0 & 0 & 0 \\ 0 & 2.9 & 0 & 0 \\ 0 & 0 & 2.84 & 1.5 \\ 0 & 0 & 1.5 & 1.3 \end{pmatrix},$$

Q_1=eye(4,4)/2; Q_1(1,1)=2.0; Q_1(4,4)=1.5;
$Q_2 = 0.5 * Q_1$;

$R_{11} = -1.909 \in\in \mathbf{R}^{1\times 1}$;
R_{22} = -eye(4,4); R_{22}(1,1)=-50; R_{22}(4,4)=-30;

Both algorithms use the stop criteria $\|\mathcal{R}(\mathcal{X}^{(k)})\| \le tol = 1.0e-14$. We execute algorithms for different values of γ and compare results. The values of γ are: $\gamma = -5$, $\gamma = -3$, $\gamma = -1$ and $\gamma = -0.75$. Table 1 involves the results for different values of γ. The average CPU time (avCPU) is available for 100 runs. Additional computational advantages of (6)-(9) are (a) the algorithm easy to extend for parallel computations; (b) the algorithm computes only one inverse matrix, that is $(\gamma I_n + A)^{-1}$, which is an $n \times n$ matrix.

According to Table 1, we have seen the both iterative methods need the same number of iteration steps for $\gamma = -1$ and $\gamma = -0.75$. Moreover, the DMLNM is faster than the LNM for all values of γ. Yet, the conclusion is that the DMLNM iteration is faster than LNM and effective one for small values of $|\gamma|$. More computational experiments can be found in [7].

Table 1. Comparison of LNM and DNLNM for minimal solution of $\mathcal{R}(\mathcal{X}) = 0$.

	LNM (2)-(3)		DMLNM (6)-(9)	
γ	avIt	avCPU seconds	avIt	avCPU seconds
-5	43	0.063	55	0.06
-3	26	0.084	37	0.047
-1	23	0.04	22	0.029
-0.75	30	0.066	30	0.035

4. CONCLUSION

The computation of the stabilizing solution of the Nash-Riccati equations is important for applications. Moreover, it is important to construct fast iterative methods to find this solution. Here, we were presented a convergence proof to effective iteration scheme (6)-(9). The computational simplicity of the algorithm lead us to the efficiency of the proposed iteration. Related discussions are expected to lead to new computational algorithms to similar problems. The fast computation of the stabilizing solution is important to construct the Nash optimal strategies for the corresponding game model and finally to find the Nash equilibrium point.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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