Iraqi Journal for Computer Science and Mathematics
Journal Homepage: http://journal.esj.edu.iq/index.php/IJCM
ISSN: 2788-7421 p-ISSN: 2958-0544

Decision-making techniques based on similarity measures of possibility interval fuzzy soft environment

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DOI: https://doi.org/10.52866/ijcsm.2023.04.04.003
Received April 2023; Accepted September 2023; Available online October 2023

ABSTRACT: The fuzzy set (FS) is a powerful logic designed to measure uncertain information and plays a crucial role in solving real-life problems such as decision-making. This article is split into two parts. In the first part, we highlight the notion of possibility for interval-valued fuzzy soft sets (PI-VFSSs) by collecting all interval fuzzy sets (IVFSs) with soft sets (SSs) under possibility properties. A basic set theories, some numerical examples, and some properties for this hybrid model have been created. Then, in order to test the efficiency of the proposed method in solving some real-life problems, we construct an algorithm using a PIV-FSS. On the other hand, in the second part, we apply similarity techniques to the proposed model through the analysis of similarity measures of two PIV-FSSs. After that, a new algorithm based on these measures is applied to the medical diagnosis to find out whether or not a patient has a respiratory disease. Finally, some theories that show the relationship between PIV-FSS with some mathematical operations have been proposed.

Keywords: Fuzzy set; soft set; interval-fuzzy set;similarity measure and its applications; possibility interval-fuzzy soft set

1. INTRODUCTION

In our daily lives, we face many situations and difficulties that contain tremendous uncertainty. To overcome these difficulties, scientists around the world set out to devise mathematical tools that could deal accurately with these difficulties. The first of those scholars was Zadeh [1] who he invented the idea of the fuzzy set theory (FS) as an extension of the crisp set. Zadeh characterizes FS as a single membership function degree fall, with a range in the unit interval [0, 1]. In some life situations, the user finds it difficult to deal with the uncertainty of the singular framework. For example, when someone is asked about the expected temperature for tomorrow here, this person will find it difficult to specify his answer in the singular frame, but at the same time, he will find greater flexibility and ease when asking him to specify his answer in the interval frame. This reason prompted Zadeh, once again, to redefine the concept of fuzzy sets in an interval form when he defined the notion of interval valued-FS (IVFS) [2]. At present, FS and IFS have been expanded by researchers into many concepts and ideas, such as intuitionistic fuzzy sets [3], interval-valued intuitionistic fuzzy sets [4], neutrosophic sets [5], interval neutrosophic sets [6] and these ideas are employed successfully to handle uncertainty issues in several areas of application, such as decision making, control, medical diagnosis, game theory, and computer vision [7]–[16]. The point of difference between these structures lies in their mathematical structure, i.e., in the number and function of their membership functions. For example, intuitionistic fuzzy sets contain two membership functions, while fuzzy sets contain one membership function, but neutrosophic sets contain three membership functions, and all of them have codomains belong the interval [0, 1].

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As the passing of the years progressed, one of the Russian scientists indicated that there is a scientific gap in these concepts mentioned above, as they indicated that these concepts do not contain sufficient tools to deal with parameters. To overcome this gap, Molodtsov [17] proposed a new parameter tool called soft set (SS) to handle the uncertainty in parameter structure. The merger of both the SS and the FS generated many contributions by researchers; for example, Cagman et al. [18] defined some related properties of the FS-soft set (FSS). Tanay and Kandemir [19] introduced some topological structures on FSS, such as the neighbourhood of a FSS, interior FSS, and FSS subspace topology. Majumdar and Samanta [20] examined several similarity measures of FSS. Al-Quran et al.[21, 22] utilized both bipolarity and complexity properties to handle some limitations in DM problems. Jiang et al. [23] initiated the idea of the IVF-soft set and applied some entropy measures to it. Al-Sharqi et al. [24]–[28] extended the IVFSS from real space to complex space and applied these extensions to solve some real-life problems.

On the other hand, Alkhazaleh et al.[29] established the concept of probability FSSs, and they also introduced the idea of applying some properties of probability to the fuzzy theory. From a scientific view of the concept of Alkhazaleh et al., we note that the novelty of this concept is to provide decision-makers with an evaluation score that belongs to the period [0,1]. In this work, we will develop the idea of probability IV-FSS (PIV-FSS) by providing an evaluation score that belongs to the period [0, 1] for all IV-FSS environments.
In addition, similarity techniques are considered effective tools for measuring the percentage of similarity between two things or objects in a given environment; therefore, we will apply these techniques to our proposed model to calculate the percentage of similarity between two PIV-FSSs.

This paper is split into six parts, which are as follows: We revisit some critical definitions and properties of our idea in Section 2. The general framework of PIV-FSS, some properties, and numerical examples are shown in Section 3. Following Sections 2 and 3, in Section 4, we present necessary operations on the PIV-FSS jointly with some propositions and numerical examples. One application in decision-making is solved by PIV-FSS-setting in Section 5. In Section 6, we define the similarity measure between two PIV-FSSs and show the importance of this measure in medical diagnosis. Finally, the brief conclusions of this work are presented in Section 7.

2. PRELIMINARIES

In this part of the article, some critical definitions and properties of our idea like FS, IVFS, and SS, are given.

Definition 1. *[1] An FS $\ddot{\mathcal{n}}$ is characterized by $\ddot{\mathcal{n}} = \left\{\left\langle v, \dot{\mathcal{F}}_{\ddot{\mathcal{n}}}(v), \forall v \in \mathcal{V}\right\rangle\right\}$ such that $\dot{\mathcal{F}}_{\ddot{\mathcal{n}}}(v) : \mathcal{V} \rightarrow [0,1]$ is real-valued truth-membership.*

Definition 2. *[30] An IVFS $\ddot{\mathcal{n}}$ is characterized by $\ddot{\mathcal{n}} = \left\{\left\langle v, \dot{\mathcal{F}}^{l}{}_{\ddot{\mathcal{n}}}(v), \dot{\mathcal{F}}^{u}{}_{\ddot{\mathcal{n}}}(v) \forall v \in \mathcal{V}\right\rangle\right\}$ such that $\dot{\mathcal{F}}^{l}{}_{\ddot{\mathcal{n}}}(v), \dot{\mathcal{F}}^{u}{}_{\ddot{\mathcal{n}}}(v) : \mathcal{V} \rightarrow [0,1]$ are real-valued lower and upper truth-membership.*

Definition 3. *[30]* ***(Properties of IV-FS)***

Let $\ddot{\mathcal{n}}_1 = \left\{\langle v, \left[\dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_1}(v), \dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_1}(v)\right]\rangle v \in \mathcal{V}\right\}$ and $\ddot{\mathcal{n}}_2 = \left\{\langle v, \left[\dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_2}(v), \dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_2}(v)\right]\rangle v \in \mathcal{V}\right\}$ be two IVFSs. Then the following basic operation on IVFSs is defined as for all $v \in \mathcal{V}$:

i. $\ddot{\mathcal{n}}_1 \subseteq \ddot{\mathcal{n}}_2$ *if and only if* $\dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_1}(v) \leq \dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_2}(v)$ *and* $\dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_1}(v) \leq \dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_2}(v)$.

ii. $\ddot{\mathcal{n}}_1 = \ddot{\mathcal{n}}_2$ *if and only if* $\dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_1}(v) = \dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_2}(v)$ *and* $\dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_1}(v) = \dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_2}(v)$.

iii. The complement of $\ddot{\mathcal{n}}_1$ *denotes* $\ddot{\mathcal{n}}_1^c$*, such that* $\dot{\mathcal{F}}_{\ddot{\mathcal{n}}_1^c}(v) = \left[1 - \dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_1}(v), 1 - \dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_1}(v)\right]$.

iv. If $\ddot{\mathcal{n}}_3 = \ddot{\mathcal{n}}_1 \cup \ddot{\mathcal{n}}_2$ *then the* $\dot{\mathcal{F}}_{\ddot{\mathcal{n}}_3}(v) = max\{\dot{\mathcal{F}}_{\ddot{\mathcal{n}}_1}(v), \dot{\mathcal{F}}_{\ddot{\mathcal{n}}_2}(v)\} = [max[\dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_1}(v), \dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_2}(v)],$
$max[\dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_1}(v), \dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_2}(v)] = \dot{\mathcal{F}}_{\ddot{\mathcal{n}}_1}(v) \vee \dot{\mathcal{F}}_{\ddot{\mathcal{n}}_2}(v)$.

v. $\ddot{\mathcal{n}}_3 = \ddot{\mathcal{n}}_1 \cap \ddot{\mathcal{n}}_2$ *then the* $\dot{\mathcal{F}}_{\ddot{\mathcal{n}}_3}(v) = max\{\dot{\mathcal{F}}_{\ddot{\mathcal{n}}_1}(v), \dot{\mathcal{F}}_{\ddot{\mathcal{n}}_2}(v)\} = [max[\dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_1}(v), \dot{\mathcal{F}}^{l}_{\ddot{\mathcal{n}}_2}(v)], max[\dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_1}(v), \dot{\mathcal{F}}^{u}_{\ddot{\mathcal{n}}_2}(v)] = \dot{\mathcal{F}}_{\ddot{\mathcal{n}}_1}(v) \wedge \dot{\mathcal{F}}_{\ddot{\mathcal{n}}_2}(v)$.

Definition 4. *[17] A pair $\left(\tilde{\mathcal{F}}, \mathcal{E}\right)$ is a SS on fixed set $\mathcal{V}$, where $\tilde{\mathcal{F}} : \mathcal{E} \rightarrow \mathcal{P}(\mathcal{V})$ such that $\mathcal{A}$ is a subset of attributes set $\mathcal{E}$.*

3. POSSIBILITY INTERVAL- VALUED FUZZY SOFT SETS(PI-VFSSS)

In this part of the article, the main definitions and properties of our model are given, as well as some numerical examples to illustrate these ideas.

Definition 5. *The order pair* $\left(\mathcal{F}_\eta, \mathcal{Z}\right)$ *is called the possibility interval-fuzzy soft set (PI-FS-set) over a nonempty soft universe* $(\mathcal{V}, \mathcal{Z})$ *if*

$$\mathcal{F}_\eta : A \to IF^{\mathcal{V}} \times I^{\mathcal{V}}$$

defined by

$$\mathcal{F}_\eta(\tau_i) = \{\mathcal{F}(\tau_i)(v_n), \eta(\tau_i)(v_n)\}$$

with

$$\mathcal{F}(\tau_i)(v_n) = \left\langle \rho^l(\tau_i)(v_n), \rho^u(\tau_i)(v_n)\right\rangle \forall \tau_i \in \mathfrak{P} \subseteq \mathcal{Z}, v_n \in \mathcal{V}.$$

Where,

1. *For* $\mathcal{V} = \{v_1, v_2, v_3, ..., v_n\}$*be a non-empty initial universe,* $\mathfrak{P} = \left\{\tau_1, \tau_2, \tau_3, ..., \tau_j\right\}$*be a parameters set.*
2. $\mathcal{F} : \mathcal{Z} \to I - F^{\mathcal{V}}$ *and* $\eta : \mathcal{Z} \to I^{\mathcal{V}}$*,*$I - F^{\mathcal{V}}$ *and* $I^{\mathcal{V}}$ *indicates the collection of all interval-fuzzy set and fuzzy subset of* $\mathcal{V}$ *respectively.*
3. $\mathcal{F}(\tau)(v_n)$ *is the degree of interval-fuzzy membership of* $v \in \mathcal{V}$ *in* $\mathcal{F}(\tau)$*, i.e(*$\rho^l(\tau)(v_n), \rho^u(\tau)(v_n)$*) denotes to lower and upper bounded of interval-fuzzy memberships receptively.*
4. $\eta(\tau)(v_n)$ *is a degree of possibility membership of* $v \in \mathcal{V}$ *in* $\mathcal{F}(\tau)$*.*

so $\mathcal{F}_\eta(z_i)$ can be written as below:

$$\left\{\left(\frac{v_1}{\mathcal{F}(\tau)(v_1)}, \eta(\tau)(v_1)\right), \left(\frac{v_2}{\mathcal{F}(\tau)(v_2)}, \eta(\tau)(v_2)\right), \left(\frac{v_3}{\mathcal{F}(\tau)(v_3)}, \eta(\tau)(v_3)\right), ..., \left(\frac{v_n}{\mathcal{F}(\tau)(v_n)}, \mu(\tau)(v_n)\right)\right\}$$

for $i = 1, 2, 3, ..., n$

Remark 1. *: If we have* $\mathcal{P} \subseteq \mathcal{Z}$ *it is also possible to write a PI-FSs as* $\left(\mathfrak{F}_\eta, \mathcal{P}\right)$ *and to essay way the PI-FSs can be written as* $\mathcal{F}_\eta$*.*

Example 1. *Let* $\mathcal{V} = \{v_1, v_2, v_3\}$*be the universal set of elements contains three houses, let* $\mathcal{P} = \{\tau_1, \tau_2, \tau_3, \tau_4\}$*be a parameters set (features of these houses), where* τ_1 *=Cheap,* τ_2 *=Beautiful,*τ_3 *=Size,*τ_4 *=Expensive. Assume that* $\mathcal{F}_\eta : A \to I - F^{\mathcal{V}} \times I^{\mathcal{V}}$ *is a PI-FSSs function represented as follows:*

$$\mathcal{F}_\eta(\tau_1) = \left\{\left(\frac{v_1}{\langle[0.5,0.3]\rangle}, 0.2\right), \left(\frac{v_2}{\langle[0.4,0.2]\rangle}, 0.3\right), \left(\frac{v_3}{\langle[0.6,0.1]\rangle}, 0.5\right)\right\}$$
$$\mathcal{F}_\eta(\tau_2) = \left\{\left(\frac{v_1}{\langle[0.6,0.3]\rangle}, 0.5\right), \left(\frac{v_2}{\langle[0.1,0.3]\rangle}, 0.7\right), \left(\frac{v_3}{\langle[0.9,0.4]\rangle}, 0.6\right)\right\}$$
$$\mathcal{F}_\eta(\tau_3) = \left\{\left(\frac{v_1}{\langle[0.3,0.4]\rangle}, 0.1\right), \left(\frac{v_2}{\langle[0.4,0.5]\rangle}, 0.3\right), \left(\frac{v_3}{\langle[0.2,0.4]\rangle}, 0.4\right)\right\}$$
$$\mathcal{F}_\eta(\tau_4) = \left\{\left(\frac{v_1}{\langle[0.1,0.1]\rangle}, 0.4\right), \left(\frac{v_2}{\langle[0.6,0.4]\rangle}, 0.2\right), \left(\frac{v_3}{\langle[0.3,0.2]\rangle}, 0.5\right)\right\}$$

Now, we can present PI-FSSs $\left(\mathcal{F}_\eta, \mathcal{P}\right)$ *as be formed of the following aggregate of approximations:*

$$\left(\mathcal{F}_\eta, \mathcal{P}\right)$$
$$\left\{\mathcal{F}_\eta(\tau_1) = \left\{\left(\frac{v_1}{\langle[0.5,0.3]\rangle}, 0.2\right), \left(\frac{v_2}{\langle[0.4,0.2]\rangle}, 0.3\right), \left(\frac{v_3}{\langle[0.6,0.1]\rangle}, 0.5\right)\right\}\right.$$
$$\mathcal{F}_\eta(\tau_2) = \left\{\left(\frac{v_1}{\langle[0.6,0.3]\rangle}, 0.5\right), \left(\frac{v_2}{\langle[0.1,0.3]\rangle}, 0.7\right), \left(\frac{v_3}{\langle[0.9,0.4]\rangle}, 0.6\right)\right\}$$
$$\mathcal{F}_\eta(\tau_3) = \left\{\left(\frac{v_1}{\langle[0.3,0.4]\rangle}, 0.1\right), \left(\frac{v_2}{\langle[0.4,0.5]\rangle}, 0.3\right), \left(\frac{v_3}{\langle[0.2,0.4]\rangle}, 0.4\right)\right\}$$
$$\left.\mathcal{F}_\eta(\tau_4) = \left\{\left(\frac{v_1}{\langle[0.1,0.1]\rangle}, 0.4\right), \left(\frac{v_2}{\langle[0.6,0.4]\rangle}, 0.2\right), \left(\frac{v_3}{\langle[0.3,0.2]\rangle}, 0.5\right)\right\}\right\}$$

Then we say that $\left(\mathcal{F}_\eta, \mathcal{P}\right)$ *is a is said to be possibility interval-fuzzy soft set (P-IFSSs) over soft universe* $(\mathcal{V}, \mathcal{Z})$*.*

Definition 6. *For two PI-FSSs* $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ *and* $\left(\mathcal{G}_\varphi, \mathcal{B}\right)$ *over* $(\mathcal{V}, \mathcal{Z})$*.Then* $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ *is said to be be a PI-FS-subset of* $\left(\mathcal{G}_\varphi, \mathcal{B}\right)$ *if* $\mathcal{A} \subseteq \mathcal{B}$*, and* $\forall \tau \in \mathcal{A} \subseteq \mathcal{Z}$ *the next conditions are fulfilled:*

1. $\eta(\tau)$ *is fuzzy subset of* $\varphi(\tau)$*.*

2. $\mathcal{F}_\eta(\tau)$ is interval fuzzy subset of $\mathfrak{G}_\varphi(\tau)$.
And we denoted this relation as $\left(\mathcal{F}_\eta, \mathcal{A}\right) \subseteq \left(\mathcal{G}_\varphi, \mathcal{B}\right)$.In this issue, $\left(\mathfrak{G}_\varphi, \mathcal{B}\right)$ is named a PI-FS-superset of $\left(\mathcal{F}_\eta, \mathcal{A}\right)$.

Definition 7. *If $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ and $\left(\mathcal{G}_\varphi, \mathcal{B}\right)$ be two PI-FSSs over $(\mathcal{V}, \mathcal{Z})$.Then $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ is equal to $\left(\mathcal{G}_\varphi, \mathcal{B}\right)$ if $\forall \tau \in \mathcal{A} \subseteq \mathcal{Z}$ the next conditions are fulfilled:*

1. $\eta(\tau)$ is equal of $\varphi(\tau)$.

2. $\mathcal{F}_\eta(\tau)$ is equal of $\mathcal{G}_\varphi(\tau)$.
And we denoted this relation as $\left(\mathcal{F}_\eta, \mathcal{A}\right) = \left(\mathcal{G}_\varphi, \mathcal{B}\right)$.In this words, $\left(\mathcal{G}_\varphi, \mathcal{B}\right)$ is equal of $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ if $\left(\mathfrak{G}_\varphi, \mathcal{B}\right)$ is PI-FS-subset of $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ and $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ is PI-FS-subset of $\left(\mathcal{G}_\varphi, \mathcal{B}\right)$.

Definition 8. *A PI-FSS $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ is named null-PI-FSS, indicated by $\left(\ddot{\Phi}_\eta, \mathcal{A}\right)$ and given as follows*

$$\ddot{\Phi}_\eta(\tau_i) = \{\mathcal{F}(\tau_i)(\nu_n), \eta(\tau_i)(\nu_n)\}, \ \forall \tau_i \in A \subseteq \mathcal{Z}$$

where $\mathcal{F}(\tau_i)(\nu_n) = \langle[0,0]\rangle$ such that $\forall \tau_i \in A \subseteq \mathcal{Z}, \nu \in \mathcal{V}$ we have $\rho^l(\tau_i)(\nu_n) = 0, \rho^u(\tau_i)(\nu_n) = 0$ and $\eta(\tau_i)(\nu_n) = 0$.

Definition 9. *A PI-FSS $(\mathcal{F}, \mathcal{A})$ is named to be absolute-PI-FSS, indicate by $\left(\mathcal{F}_\eta, \mathcal{A}\right)_{Abso}$ and given as follows*

$$\mathfrak{F}_\eta(\tau_i) = \{\mathfrak{F}(\tau_i)(\nu_n), \mu(\tau_i)(\nu_n)\}, \ \forall \tau_i \in A \subseteq \mathcal{Z}$$

where $\mathcal{F}(\tau_i)(\nu_n) = \langle[1,1]\rangle$ such that $\forall \tau_i \in \mathcal{A} \subseteq \mathcal{Z}, \nu \in \mathcal{V}$ we have $\rho^l(\tau_i)(\nu_n) = 1, \rho^u(\tau_i)(\nu_n) = 1$ and $\eta(\tau_i)(\nu_n) = 1$.

4. FUNDAMENTAL SET-THEORETIC OPERATIONS OF PI-VFSS

In the next part, some fundamental mathematical operations on PI-FSS will be discussed, namely complement on one set of PI-VFSS, union, and intersection on two or more sets of PI-VFSS, followed by AND and OR operations on two or more sets of PI-VFSS. Finally, this part offers some properties related to these operations with suitable examples.

Definition 10. *Let $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ be a PI-FSS over fixed set (soft universe) $(\mathcal{V}, \mathcal{Z})$. Then the complement of a PI-FSS $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ indicated by $\left(\mathcal{F}_\eta, \mathcal{A}\right)^c$ is given as follows:*

$$\left(\mathcal{F}_\eta, \mathcal{A}\right)^c = \mathcal{F}_\eta^c(\tau_i) = \{\ddot{c}(\mathcal{F}(\tau)(\nu_n)), \dot{c}(\mu(\tau)(\nu_n))\}$$

where $\ddot{c}$ indicates a interval fuzzy complement and $\dot{c}$ indicates a fuzzy complement.

Example 2. *Take the part given in Example 1. where*
$$\mathcal{F}_\eta(\tau_1) = \left\{\left(\frac{\nu_1}{\langle[0.5,0.6]\rangle}, 0.2\right), \left(\frac{\nu_2}{\langle[0.4,0.7]\rangle}, 0.3\right), \left(\frac{\nu_3}{\langle[0.6,0.6]\rangle}, 0.8\right)\right\}$$

Now, by employing the interval-fuzzy complement and fuzzy complement, we get the complement of the part that is given by
$$\mathcal{F}_\eta^c(\tau_1) = \left\{\left(\frac{\nu_1}{\langle[0.4,0.5]\rangle}, 0.8\right), \left(\frac{\nu_2}{\langle[0.3,0.6]\rangle}, 0.7\right), \left(\frac{\nu_3}{\langle[0.4,0.4]\rangle}, 0.2\right)\right\}$$

Proposition 1. *Let $\left(\mathcal{F}_\eta, \mathfrak{A}\right)$ be a PI-FSS over fixed set $(\mathcal{V}, \mathcal{Z})$.Then the following property applies:*

$$\left(\left(\mathcal{F}_\eta, \mathcal{A}\right)^c\right)^c = \left(\mathcal{F}_\eta, \mathcal{A}\right)$$

Proof. Assume that $\left(\mathcal{F}_\eta, \mathcal{A}\right)$ be a PI-FSS over fixed set $(\mathcal{V}, \mathcal{Z})$ and defined as $\left(\mathcal{F}_\eta, \mathcal{A}\right) = \mathcal{F}_\eta(\tau_i) = (\mathcal{F}(\tau_i), \eta(\tau_i))$.
Now, let $\left(\mathcal{F}_\eta, \dot{A}\right)^c = \left(\mathcal{G}_\varphi, \mathcal{A}\right)$.
Then based on definition 8 $\left(\mathfrak{G}_\varphi, \mathcal{B}\right) = \mathcal{G}_\varphi(\tau_i) = (\mathcal{G}(\tau_i), \varphi(\tau_i))$.Such that $\mathcal{G}(\tau_i) = \ddot{c}(\mathcal{F}(\tau_i))$ and $\varphi(\tau_i) = \dot{c}(\eta(\tau_i))$. Thus it leads us to
$\left(\mathcal{G}_\eta, \mathcal{B}\right)^c = \mathcal{G}_\varphi^c(\tau_i) = (\ddot{c}(\mathcal{G}(\tau_i)), \dot{c}(\varphi(\tau_i))) = (\ddot{c}(\ddot{c}(\mathcal{F}(\tau_i))), \dot{c}(\dot{c}(\eta(\tau_i)))) = (\mathcal{F}(\tau_i), \eta(\tau_i)) = \left(\mathcal{F}_\eta, \mathcal{A}\right)$.
Thus $\left(\left(\mathcal{F}_\eta, \mathcal{A}\right)^c\right)^c = \left(\mathcal{G}_\varphi, \mathcal{B}\right)^c = \left(\mathcal{F}_\eta, \mathcal{A}\right)$. Hence we get $\left(\left(\mathcal{F}_\eta, \mathcal{A}\right)^c\right)^c = \left(\mathcal{F}_\eta, \mathcal{A}\right)$.

Definition 11. *If $\left(\mathcal{F}_\eta,\mathcal{A}\right)$ and $\left(\mathcal{G}_\varphi,\mathcal{A}\right)$ two PI-FSSs on fixed set (soft universe) $(\mathcal{V},\mathcal{Z})$. Then the union operation of these sets is also PI-FSS $(\mathcal{H}_\Psi,\mathcal{C})$ and denoted by $(\mathcal{H}_\Psi,\mathcal{C}) =\left(\mathcal{F}_\eta,\mathcal{A}\right) = \ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{B}\right)$. Where $\dot{C} = \dot{A}\cup\dot{B}$ and*
$\Psi(\tau_i) = max(\eta(\tau_i),\varphi(\tau_i)), \qquad \forall\tau_i \in \dot{C} \subseteq \mathcal{Z} = \{\mathcal{P}\}.$
$\mathcal{H}(\eta_i) = \mathcal{F}(\tau_i)\ddot{\cup}\mathcal{G}(\tau_i), \qquad \forall\tau_i \in \dot{C} \subseteq \mathcal{Z} = \{\mathcal{P}\}.$

where
$$\mathcal{H}(\tau_i) = \begin{cases} \mathcal{F}(\tau_i) & ,if\tau_i \in \mathcal{A}-\mathcal{B} \\ \mathcal{G}(\tau_i) & ,if\tau_i \in \mathcal{B}-\mathcal{A} \\ max(\mathcal{F}(\tau_i),\mathcal{G}(\tau_i)) & ,if\tau_i \in \mathcal{A}\cap\mathcal{B} \end{cases}$$

Proposition 2. *Let $\left(\mathcal{F}_\eta,\mathcal{A}\right)$, $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ and $(\mathcal{H}_\Psi,\mathcal{C})$ be any three optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then the following results are achieved:*

(i). $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\mathcal{G}_\varphi,\dot{B}\right) =\left(G_\varphi,\mathcal{B}\right)\ddot{\cup}\left(\mathcal{F}_\mu,\mathcal{A}\right)$.*(Commutative Condition)*

(ii) $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\cup}(\mathcal{H}_\Psi,\mathcal{C})\right) =\left(\left(\mathcal{F}_\mu,\mathcal{A}\right)\ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)\ddot{\cup}(\mathcal{H}_\Psi,\mathcal{C})$.*(Associative Condition)*

Proof. Assume that $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{B}\right) =(\mathcal{H}_\Psi,\mathcal{C})$.Then based on Definition 9, $\forall\tau_i \in \mathcal{C} \subseteq \mathcal{P} = \{\mathcal{Z}\}$. we have
$$(\mathcal{H}_\Psi,\mathcal{C}) = \mathcal{H}_\Psi(\tau_i) = (\mathcal{H}(\tau_i),\Psi(\tau_i))$$
where $\mathcal{H}(\tau_i) = \mathcal{F}(\tau_i)\ddot{\cup}\mathcal{G}(\tau_i)$ and $\Psi(\tau_i) = max(\eta(\tau_i),\varphi(\tau_i))$. So,$\mathcal{H}(\tau_i) = \mathcal{F}(\tau_i)\ddot{\cup}\mathcal{G}(\tau_i) = \mathcal{G}(\tau_i)\ddot{\cup}\mathcal{F}(\tau_i)$ and $\Psi(\tau_i) = max(\eta(\tau_i),\varphi(\tau_i)) = max(\varphi(\tau_i),\eta(\tau_i))$. we have the union of these sets is commutative by Definition 9.
Therefore,$(\mathcal{H}_\Psi,\mathcal{C}) =\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\cup}\left(\mathcal{F}_\varphi,\mathcal{A}\right)$.
Then we get the union of two PNSE-sets is commutative, such that $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\mathcal{G}_\varphi,\dot{B}\right) =\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\cup}\left(\mathcal{F}_\eta,\mathcal{A}\right)$.
(ii) The proof of this part is equivalent to (i) and is therefore overlooked.

Definition 12. *If $\left(\mathcal{F}_\eta,\mathcal{A}\right)$ and $\left(\mathcal{G}_\varphi,\mathcal{A}\right)$ two PI-FSSs on fixed set (soft universe) $(\mathcal{V},\mathcal{Z})$. Then the intersection operation of these sets is also PI-FSS $(\mathcal{H}_\Psi,\mathcal{C})$ and denoted by $(\mathcal{H}_\Psi,\mathcal{C}) =\left(\mathcal{F}_\eta,\mathcal{A}\right) = \ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{B}\right)$. Where $\mathcal{C} = \mathcal{A}\cup\mathcal{B}$ and*
$\Psi(\tau_i) = min(\eta(\tau_i),\varphi(\tau_i)), \qquad \forall\tau_i \in \mathcal{C} \subseteq \mathcal{P} = \{\mathcal{Z}\}.$
$\mathcal{H}(\eta_i) = \mathcal{F}(\tau_i)\ddot{\cap}\mathcal{G}(\tau_i), \qquad \forall\tau_i \in \dot{C} \subseteq \mathcal{P} = \{\mathcal{Z}\}.$

where
$$\mathcal{H}(\tau_i) = \begin{cases} \mathcal{F}(\tau_i) & ,if\tau_i \in \mathcal{A}-\mathcal{B} \\ \mathcal{G}(\tau_i) & ,if\tau_i \in \mathcal{B}-\mathcal{A} \\ max(\mathcal{F}(\tau_i),\mathcal{G}(\tau_i)) & ,if\tau_i \in \mathcal{A}\cap\mathcal{B} \end{cases}$$

Proposition 3. *Let $\left(\mathcal{F}_\eta,\mathcal{A}\right)$, $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ and $(\mathcal{H}_\Psi,\mathcal{C})$ be any three optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then the following results are achieved:*

(i). $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{B}\right) =\left(G_\varphi,\mathcal{B}\right)\ddot{\cap}\left(\mathcal{F}_\eta,\mathcal{A}\right)$.*(Aommutative Condition)*

(ii) $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\cap}(\mathcal{H}_\Psi,\mathcal{C})\right) =\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)\ddot{\cup}(\mathcal{H}_\Psi,\mathcal{C})$.*(Associative Condition)*

Proof. The proof of these two parts (i, ii) is equivalent to (i, ii) in proposition 1 is overlooked.

Proposition 4. *Let $\left(\mathcal{F}_\eta,\mathcal{A}\right)$, $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ and $(\mathcal{H}_\Psi,\mathcal{C})$ be any three optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then the following results are satisfying:*
(i). $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\cap}(\mathcal{H}_\Psi,\mathcal{C})\right) =\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)\ddot{\cap}\left(\left(\mathcal{F}_\eta,A\right)\ddot{\cup}(\mathcal{H}_\Psi,\mathcal{C})\right)$

(ii). $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\cup}(\mathcal{H}_\Psi,\mathcal{C})\right) =\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)\ddot{\cup}\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}(\mathcal{H}_\Psi,\mathcal{C})\right)$

Proof. The proof of these propositions clear dependency Definitions 9 and 10 and is therefore overlooked.

Proposition 5. *Let $\left(\mathcal{F}_\eta,\mathcal{A}\right)$ and $\left(\mathcal{G}_\varphi,\mathcal{A}\right)$ be any two optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then De Morgan's laws satisfying:*
(i). $\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{A}\right)\right)^c =\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)^c\ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{A}\right)^c\right)$.

(ii). $\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{A}\right)\right)^c=\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)^c\ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{A}\right)^c\right)$.

Proof. (i) Assume that $\left(\mathcal{F}_\eta,\mathcal{A}\right)$ and $\left(\mathcal{G}_\varphi,\mathcal{A}\right)$ be any two optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$ defined as following:

$$(\mathcal{F}_\Psi,\mathcal{A})=\mathcal{F}_\eta(\tau_i)=(\mathcal{F}(\tau_i),\eta(\tau_i)),\quad \forall\tau_i\in\mathcal{C}\subseteq\mathcal{Z}.$$
$$(\mathcal{G}_\Psi,\mathcal{B})=\mathcal{G}_\eta(\tau_i)=(\mathcal{G}(\tau_i),\eta(\tau_i)),\quad \forall\tau_i\in\mathcal{C}\subseteq\mathcal{Z}.$$

Now, since the commutative and associative properties are fulfilled with PI-FSS, it follows that
$\left(\mathcal{F}_\eta,\mathcal{A}\right)^c\ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{B}\right)^c$
$=(\mathcal{F}(z_i),\eta(\tau_i))^c\ddot{\cup}(\mathcal{G}(\tau_i),\varphi(\tau_i))^c$
$=(\ddot{c}(\mathcal{F}(\tau_i)),c(\eta(\tau_i)))\ddot{\cup}(\ddot{c}(\mathcal{G}(\tau_i)),\dot{c}(\varphi(\tau_i)))$
$=(\ddot{c}(\mathcal{F}(\tau_i)),\ddot{\cup}\ddot{c}(\mathcal{G}(\tau_i)))\max(\dot{c}(\eta(\tau_i)),\dot{c}(\varphi(\tau_i)))$
$=(\ddot{c}(\mathcal{F}(\tau_i)\ddot{\cap}\mathcal{G}(\tau_i))),\dot{c}(\min(\eta(\tau_i),\varphi(\tau_i)))$
$=\left((\mathcal{F}_\tau,\mathcal{A})\ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)^c$.

(ii) (ii)The proof of the(ii) is comparable to the proof of the (i) and therefore overlooked.

Definition 13. *Let $\left(\mathcal{F}_\eta,\mathcal{A}\right)$and $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ be any two optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then $\left(\mathcal{F}_\eta,\mathcal{A}\right)$AND $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ indicated by $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ is a PI-FSS and defined as:*

$$\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)=(\mathcal{H}_\Psi,\mathcal{A}\times\mathcal{B})$$

where $(\mathcal{H}_\Psi,\mathcal{A}\times\mathcal{B})=\left(\mathcal{H}(\tau_i,\tau_j),\Psi(\tau_i,\tau_j)\right)$, such that $\mathcal{H}(\tau_i,\tau_j)=\mathcal{F}(\tau_i)\ddot{\cap}\mathcal{G}(\tau_j)$ and $\Psi(\tau_i,\tau_j)=min(\eta(z_i),\varphi(\tau_j))$, $\forall(\tau_i,\tau_j)\in$ mathcalA $\times\mathcal{B}\subseteq\mathcal{Z}and\ddot{\cap}$ depicts the basic intersection operation.

Definition 14. *Let $\left(\mathcal{F}_\eta,\mathcal{A}\right)$and $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ be any two optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then $\left(\mathcal{F}_\eta,\mathcal{A}\right)$OR $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ indicated by $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\vee}\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ is a PI-FSS and defined as:*

$$\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\vee}\left(\mathcal{G}_\varphi,\mathcal{B}\right)=(\mathcal{H}_\Psi,\mathcal{A}\times\mathcal{B})$$

where $(\mathcal{H}_\Psi,\mathcal{A}\times\mathcal{B})=\left(\mathcal{G}(\tau_i,\tau_j),\Psi(\tau_i,\tau_j)\right)$, such that $\mathcal{H}(\tau_i,\tau_j)=\mathcal{F}(\tau_i)\ddot{\cap}\mathcal{G}(\tau_j)$ and $\Psi(\tau_i,\tau_j)=min(\eta(z_i),\varphi(\tau_j))$, $\forall(\tau_i,\tau_j)\in$ mathcalA $\times\mathcal{B}\subseteq\mathcal{Z}and\ddot{\cup}$ depicts the basic union operation.

Proposition 6. *Let $\left(\mathcal{F}_\eta,\mathcal{A}\right)$ and $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ be any two optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then De Morgan's laws satisfying:*

(i). $\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\vee}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)^c=\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)^c\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)^c\right)$.

(ii). $\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)^c=\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)^c\ddot{\vee}\left(\mathcal{G}_\varphi,\mathcal{B}\right)^c\right)$.

Proof. (i) Assume that $\left(\mathcal{F}_\eta,\mathcal{A}\right)$ and $\left(\mathcal{G}_\varphi,\mathcal{B}\right)$ be any two optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$ defined as following:

$$(\mathcal{F}_\Psi,\mathcal{A})=\mathcal{F}_\eta(\tau_i)=(\mathcal{F}(\tau_i),\eta(\tau_i)),\quad \forall\tau_i\in\dot{C}\subseteq\mathcal{Z}.$$
$$\left(\mathcal{G}_\varphi,\mathcal{B}\right)=\mathcal{G}_\varphi(\tau_i)=(\mathcal{G}(\tau_i),\varphi(\tau_i)),\quad \forall\tau_i\in\dot{C}\subseteq\mathcal{Z}.$$

Now, since the commutative and associative properties are fulfilled with PI-FSS, it follows that
$\left(\mathcal{F}_\eta,\mathcal{A}\right)^c\ddot{\vee}\left(\mathcal{G}_\varphi,\mathcal{B}\right)^c$
$=(\mathcal{F}(\tau_i),\eta(\tau_i))^c\ddot{\vee}(\mathcal{G}(\tau_i),\varphi(\tau_i))^c$
$=(\ddot{c}(\mathcal{F}(\tau_i)),\dot{c}(\eta(\tau_i)))\ddot{\vee}(\ddot{c}(\mathcal{G}(\tau_i)),\dot{c}(\varphi(\tau_i)))$
$=(\ddot{c}(\mathcal{F}(\tau_i)),\ddot{\vee}\ddot{c}(\mathcal{G}(\tau_i)))\max(\dot{c}(\eta(\tau_i)),\dot{c}(\varphi(\tau_i)))$
$=(\ddot{c}(\mathcal{F}(\tau_i)\ddot{\wedge}\mathcal{G}(\tau_i))),\dot{c}(\min(\eta(\tau_i),\varphi(\tau_i)))$
$=\left(\left(\mathcal{F}_\eta,\dot{A}\right)\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)^c$.

(ii)The proof of the second part is similar to the proof of the first part therefore omitted.

Proposition 7. *Let $\left(\mathfrak{F}_\mu,A\right)$, $\left(\mathfrak{G}_\varphi,B\right)$ and $(\mathfrak{H}_\Psi,C)$ be any three optional PNSE-sets over $(\mathfrak{V},\mathfrak{Z})$.Then the following results are achieved:*

(i). $\left(\mathcal{F}_\eta,A\right)\ddot{\vee}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\vee}(\mathcal{H}_\Psi,C)\right)=\left(\left(\mathcal{F}_\eta,A\right)\ddot{\vee}\left(\mathcal{G}_\varphi,B\right)\right)\ddot{\vee}(\mathcal{H}_\Psi,\mathcal{C})$.

(ii). $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\wedge}(\mathcal{H}_\Psi,\mathcal{C})\right)=\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)\ddot{\wedge}(\mathcal{H}_\Psi,\mathcal{C})$.

(iii). $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\vee}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\wedge}(\mathcal{H}_\Psi,\mathcal{C})\right)=\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\vee}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)\ddot{\wedge}\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\vee}(\mathfrak{H}_\Psi,\mathcal{C})\right)$.

(iV). $\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\left(\mathcal{G}_\varphi,\mathcal{B}\right)\ddot{\vee}(\mathcal{H}_\Psi,\mathcal{C})\right)=\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)\right)\ddot{\vee}\left(\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}(\mathcal{H}_\Psi,\mathcal{C})\right)$.

Proof. The proof of these propositions are clear by Definitions 9 and 10 and therefore omitted.

Remark 2. *Due $\mathcal{A}\times\mathcal{B}\neq\mathcal{B}\times\mathcal{A}$, therefore "AND" operation and "OR" operation don't satisfy commutative law.*

Example 3. *Let $\left(\mathcal{F}_\eta,A\right)$ and $\left(\mathcal{G}_\varphi,B\right)$ be any two optional PI-FSSs over $(\mathcal{V},\mathcal{Z})$ and let $\mathcal{A}=\{(\tau_1),(\tau_2)\}$,$\mathcal{B}=\{(\tau_2),(\tau_1)\}$. Then the PI-FSSs are defined as bellow:*

$$\left(\mathcal{F}_\eta,\mathcal{A}\right)=$$
$$\Big\{(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.5,0.8]\rangle},0.2\Big),\Big(\tfrac{\nu_2}{\langle[0.4,0.7]\rangle},0.3\Big),\Big(\tfrac{\nu_3}{\langle[0.6,0.6]\rangle},0.5\Big)\Big\},$$
$$(\tau_2)=\Big\{\Big(\tfrac{\nu_1}{\langle[0,0.3]\rangle},0.5\Big),\Big(\tfrac{\nu_2}{\langle[0.5,0.8]\rangle},0.4\Big),\Big(\tfrac{\nu_3}{\langle[0.1,0.2]\rangle},0.9\Big)\Big\}\Big\}$$

and

$$\left(\mathcal{G}_\varphi,\mathcal{B}\right)=$$
$$\Big\{(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.3,0.4]\rangle},0.7\Big),\Big(\tfrac{\nu_2}{\langle[0.3,0.7]\rangle},0.4\Big),\Big(\tfrac{\nu_3}{\langle[0.1,0.8]\rangle},0.6\Big)\Big\},$$
$$(\tau_2)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.3,0.5]\rangle},0.8\Big),\Big(\tfrac{\nu_2}{\langle[0.6,0.9]\rangle},0.7\Big),\Big(\tfrac{\nu_3}{\langle[0.3,0.8]\rangle},1\Big)\Big\}\Big\}$$

Then,

$$\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cup}\left(\mathcal{G}_\varphi,\mathcal{B}\right)=$$
$$\Big\{(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.5,0.8]\rangle},0.7\Big),\Big(\tfrac{\nu_2}{\langle[0.4,0.7]\rangle},0.4\Big),\Big(\tfrac{\nu_3}{\langle[0.6,0.8]\rangle},0.6\Big)\Big\},$$
$$(\tau_2)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.3,0.5]\rangle},0.8\Big),\Big(\tfrac{\nu_2}{\langle[0.6,0.9]\rangle},0.7\Big),\Big(\tfrac{\nu_3}{\langle[0.3,0.8]\rangle},1\Big)\Big\}\Big\}.$$

$$\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\cap}\left(\mathcal{G}_\varphi,\mathcal{B}\right)=$$
$$\Big\{(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.3,0.4]\rangle},0.2\Big),\Big(\tfrac{\nu_2}{\langle[0.3,0.7]\rangle},0.3\Big),\Big(\tfrac{\nu_3}{\langle[0.1,0.6]\rangle},0.5\Big)\Big\},$$
$$(\tau_2)=\Big\{\Big(\tfrac{\nu_1}{\langle[0,0.3]\rangle},0.5\Big),\Big(\tfrac{\nu_2}{\langle[0.5,0.6]\rangle},0.4\Big)\ \Big(\tfrac{\nu_3}{\langle[0.1,0.2]\rangle}0,9\Big)\Big\}\Big\}.$$

$$\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\vee}\left(\mathcal{G}_\varphi,\mathcal{B}\right)=(\mathcal{H}_\Psi,\mathcal{C}=\mathcal{A}\times\mathcal{B})=$$
$$\Big\{(\tau_1),(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.5,0.8]\rangle},0.7\Big),\Big(\tfrac{\nu_2}{\langle[0.4,0.7]\rangle},0.4\Big),\Big(\tfrac{\nu_2}{\langle[0.6,0.8]\rangle},0.6\Big)\Big\},$$
$$(\tau_2),(\tau_2)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.3,0.5]\rangle},0.8\Big),\Big(\tfrac{\nu_2}{\langle[0.6,0.9]\rangle},0.7\Big),\Big(\tfrac{\nu_2}{\langle[0.3,0.8]\rangle},1\Big)\Big\},$$
$$(\tau_2),(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.3,0.4]\rangle},0.7\Big),\Big(\tfrac{\nu_2}{\langle[0.5,0.8]\rangle},0.4\Big),\Big(\tfrac{\nu_2}{\langle[0.1,0.8]\rangle},0.9\Big)\Big\}\Big\}$$

and

$$\left(\mathcal{F}_\eta,\mathcal{A}\right)\ddot{\wedge}\left(\mathcal{G}_\varphi,\mathcal{B}\right)=(\mathcal{H}_\Psi,\mathcal{C}=\mathcal{A}\times\mathcal{B})=$$
$$\Big\{(\tau_1),(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.3,0.4]\rangle},0.2\Big),\Big(\tfrac{\nu_2}{\langle[0.3,0.7]\rangle},0.3\Big),\Big(\tfrac{\nu_2}{\langle[0.1,0.6]\rangle},0.5\Big)\Big\},$$
$$(\tau_2),(\tau_2)=\Big\{\Big(\tfrac{\nu_1}{\langle[0,0.3]\rangle},0.5\Big),\Big(\tfrac{\nu_2}{\langle[0.5,0.6]\rangle},0.4\Big),\Big(\tfrac{\nu_2}{\langle[0.1,0.2]\rangle},0.9\Big)\Big\},$$
$$(\tau_2),(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0,0.3]\rangle},0.5\Big),\Big(\tfrac{\nu_2}{\langle[0.3,0.7]\rangle},0.4\Big),\Big(\tfrac{\nu_2}{\langle[0.1,0.2]\rangle},0.6\Big)\Big\}\Big\}$$

5. DECISION-MAKING APPLICATION ON PI-VFSSS

In this part of this article, a new generalized algorithm is proposed to show the efficiency of the proposed model to help the decision maker (user) make the right decision from available alternatives based on hypothetical data, as in the following example.

Example 4. *Out of three private schools, which represent by a universal set $\mathcal{V}=\{\nu_1,\nu_2,\nu_3\}$, Mr. Ahmed wanted to choose the right one for his daughter. In fact, Mr. Ahmed focused on some of the criteria or characteristics that characterize these three schools, and thus these characteristics can be represented through $\mathcal{P}=\{\tau_1,\tau_2,\tau_3\}$ where $\tau_1=teachingquality$, $\tau_2=cost$, and $\tau_3=environment$, respectively.*

Mr. Ahmed was confused about choosing the right school, so he sought help from friends (decision-makers) to help him in his choice. Now According to the evaluation of Mr. Ahmed's friend, we get the following PI-FSS.

$$\left(\mathcal{F}_\eta,\mathcal{P}\right)=$$
$$\Big\{(\tau_1)=\Big\{\Big(\tfrac{\nu_1}{\langle[0.5,0.8]\rangle},0.2\Big),\Big(\tfrac{\nu_2}{\langle[0.4,0.7]\rangle},0.3\Big),\Big(\tfrac{\nu_3}{\langle[0.6,0.6]\rangle},0.5\Big)\Big\},$$

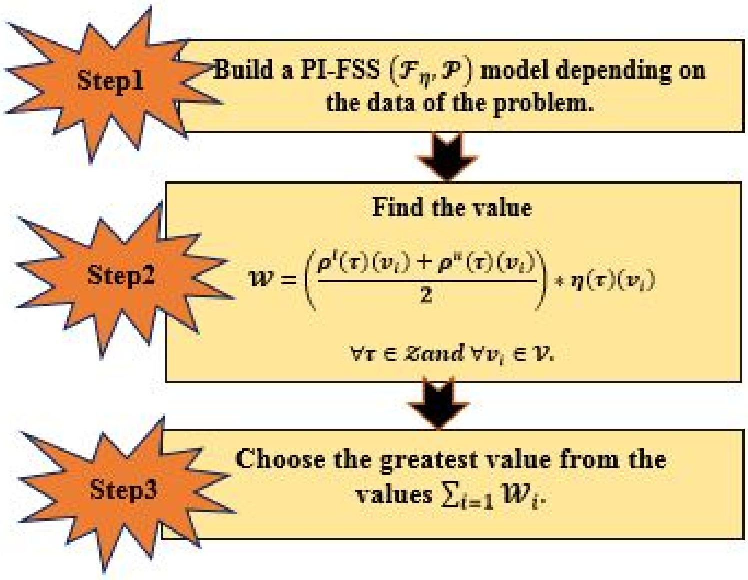

Figure 1: Algorithm 1.

Table 1. **Values of $\mathcal{W}_i$**

$\mathcal{V}_n$	v_1	v_2	v_3
(τ_1)	0.13	0.11	0.12
(τ_2)	0.38	0.15	0.33
(τ_3)	0.18	0.2	0.15
(τ_τ)	0.045	0.23	0.24
Total	0.74	0.69	0.84

$$(\tau_2) = \left\{\left(\frac{v_1}{\langle[0.6,0.9]\rangle}, 0.5\right), \left(\frac{v_2}{\langle[0.1,0.9]\rangle}, 0.3\right), \left(\frac{v_3}{\langle[0.4,0.7]\rangle}, 0.6\right)\right\},$$

$$(\tau_3) = \left\{\left(\frac{v_1}{\langle[0.4,0.8]\rangle}, 0.3\right), \left(\frac{v_2}{\langle[0.1,0.7]\rangle}, 0.5\right), \left(\frac{v_3}{\langle[0.2,0.4]\rangle}, 0.5\right)\right\},$$

$$(\tau_4) = \left\{\left(\frac{v_1}{\langle[0.3,0.6]\rangle}, 0.1\right), \left(\frac{v_2}{\langle[0.4,0.5]\rangle}, 0.5\right), \left(\frac{v_3}{\langle[0.2,0.4]\rangle}, 0.8\right)\right\}$$

To solve this problem we apply the following proposed algorithm.

Algorithm 1 which is illustrated in Figure 1.

Here Table No. 1 describes the process of interaction between the values of all the alternatives and the values of the attributes through which a full description of the problem was given. Now after applying all steps mentioned in algorithm 1, we find that the highest value is 0.84 (see Table 1), accordingly, the decision maker advises Mr. Ahmed to choose the school v_3 as the best choice among the three schools.

6. SIMILARITY MEASURE ON PI-VFSS

Similarity measures were studied by several scholars [31]–[35] to calculate the ratio of similarity between two fuzzy sets. The scholars defined several similarity measuring tools that can be used in fuzzy set theory and its extensions. These measures are employed to solve real-world problems. Now, in this part, we propose new similarity measures on PI-FSSs to calculate the ratio of similarity between two PI-FSSs.

Definition 15. *Let $\mathcal{F}_\eta$ and $\mathcal{G}_\varphi$ be two PI-FSSs over $(\mathcal{V}, \mathcal{Z})$. Then the Similarity measure between $\mathcal{F}_\eta$ and $\mathcal{G}_\varphi$ indicated by $\hat{S}\left(\mathcal{F}_\eta, \mathcal{G}_\varphi\right)$ is defined as follows:*

$$\hat{S}\left(\mathcal{F}_\eta, \mathcal{G}_\varphi\right) = \ddot{M}\left(\mathcal{F}(\tau), \mathcal{G}(\tau)\right) \times \ddot{M}\left(\eta(\tau), \varphi(\tau)\right),$$

such that

$$\ddot{M}\left(\mathcal{F}(\tau), \mathcal{G}(\tau)\right) = max\ddot{M}_i\left(\mathcal{F}(\tau), \mathcal{G}(\tau)\right),$$

$$\ddot{M}(\eta(\tau),\varphi(\tau)) = max\ddot{M}_i(\eta(\tau),\varphi(\tau)),$$

where

$$\ddot{M}_i(\mathcal{F}(\tau),\mathcal{G}(\tau)) = 1 - \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}\left(\hat{\phi}_{\mathcal{F}(\tau_i)}\left(\nu_j\right) - \hat{\phi}_{\mathcal{G}(\tau_i)}\left(\nu_j\right)\right)^2},$$

such that and,

$\hat{\phi}_{\mathcal{F}_\eta(\tau)}\left(\nu_j\right) = \frac{\rho^l{}_{\mathcal{F}\eta(\tau_i)}(\nu_j)+\rho^u{}_{\mathcal{F}\eta(\tau_i)}(\nu_j)}{2}$, $\hat{\phi}_{\mathcal{G}_\eta(\tau)}\left(\nu_j\right) = \frac{\rho^l{}_{\mathcal{G}\eta(\tau_i)}(\nu_j)+\rho^u{}_{\mathcal{G}\eta(\tau_i)}(\nu_j)}{2}$.

$$\ddot{M}(\eta(\tau_i),\varphi(\tau_i)) = 1 - \frac{\sum_{j=1}^{n}|\eta_j(\tau_i)-\varphi_j(\tau_i)|}{\sum_{j=1}^{n}|\tau_j(\tau_i)+\varphi_j(\tau_i)|}$$

Definition 16. *Let $\mathcal{F}_\eta$ and $\mathcal{G}_\varphi$ be two PI-FSSs over $(\mathcal{V},\mathcal{Z})$.We say that $\mathcal{F}_\eta$ and $\mathcal{G}_\varphi$ are significantly similar if $\hat{S}\left(\mathcal{F}_\eta,\mathcal{G}_\varphi\right) \geq \frac{1}{2}$.*

Proposition 8. *Let $\mathcal{F}_\eta$, $\mathcal{G}_\varphi$ and $\mathcal{H}_\lambda$ be three PI-FSSs over $(\mathcal{V},\mathcal{Z})$.Then the following results are achieved:*

(i). $\hat{S}\left(\mathcal{F}_\eta,\mathcal{G}_\varphi\right) = \hat{S}\left(\mathcal{G}_\eta,\mathcal{F}_\varphi\right)$.

(ii). $0 \leq \hat{S}\left(\mathcal{F}_\eta,\mathcal{G}_\varphi\right) \leq 1$.

(iii).If $\mathcal{F}_\eta = \mathcal{G}_\varphi$ *then* $\hat{S}\left(\mathcal{F}_\eta,\mathcal{G}_\varphi\right) = 1$.

(iv).$\mathcal{F}_\eta \subseteq \mathcal{G}_\varphi \subseteq \mathcal{H}_\lambda$ *then* $\hat{S}\left(\mathcal{F}_\eta,\mathcal{G}_\varphi\right) \leq \hat{S}\left(\mathcal{G}_\varphi,\mathcal{H}_\lambda\right)$.

(v).If $\mathcal{F}_\eta \cap \mathcal{G}_\varphi = \Phi \Leftrightarrow \hat{S}\left(\mathcal{F}_\eta,\mathfrak{G}_\varphi\right) = 0$.

Proof. The proof of these propositions is clear by Definition 13 and therefore omitted.

6.1 MEDICAL DIAGNOSIS APPLICATION BASED ON SIMILARITY MEASURE OF PI-VFSS

In this subsection, we create an algorithm that works to measure the similarity ratio of two PI-VFSSs. This proposed algorithm employs to estimate whether a sick person has dengue fever based on the accompanying symptoms. To run this algorithm, we created two models of PI-VFSSs depending on the assistance of the physician (expert) such that the first PI-FSS represent the illness state and the second PI-VFSS represents the ill person state. Based on similarity degree, if it is ≥ 0.5, then the ill person may have dengue fever.
Now we present the proposed algorithm 2 in this work as shown in Figure

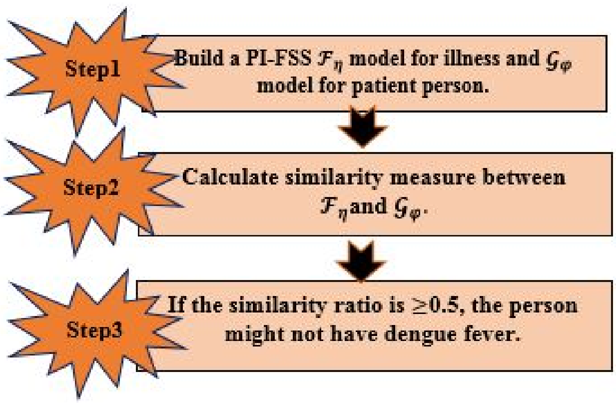

Figure 2: Algorithm 2

(Case study):

Example 5. *Let $\mathcal{V} = \{\nu_1 = ys, \nu_2 = no\}$ be a universal set containing only two alternatives, "Yes" and "No" that describe Physician opinion, and let $\mathcal{P} = \{\tau_1 = body\ temperature, \tau_2 = cough\ with\ chest\ congestion, \tau_3 = head\ ache\}$ describes attributes set that contains a set of symptoms of the illness.*

Now, we use our suggested algorithm 2 steps for the illness (dengue fever) and the patient person “Mr.Xu” as a following:
Step 1: Make the model PI-VFSS $\mathcal{F}_\eta$ for dengue fever with the assistance of one physician who met Mr. X and check him.

$$\mathcal{F}_\eta = \Big\{(\tau_1) = \Big\{\Big(\tfrac{v_1}{\langle[1,1]\rangle},1\Big),\Big(\tfrac{v_2}{\langle[1,1]\rangle},1\Big)\Big\}, (\tau_2) = \Big\{\Big(\tfrac{v_1}{\langle[1,1]\rangle},1\Big),\Big(\tfrac{v_2}{\langle[1,1]\rangle},1\Big)\Big\}, (\tau_3) = \Big\{\Big(\tfrac{v_1}{\langle[1,1]\rangle},1\Big),\Big(\tfrac{v_2}{\langle[1,1]\rangle},1\Big)\Big\}\Big\}.$$

Step 2: Build a model of PI-FSS $\mathcal{G}_\varphi$ for sick person Mr.Xu with the assistance of one physician who met Mr. Xu and checks him as following:

$$\mathcal{F}_\eta = \Big\{(\tau_1) = \Big\{\Big(\tfrac{v_1}{\langle[0.3,0.6]\rangle},0.9\Big),\Big(\tfrac{v_2}{\langle[0.5,0.4]\rangle},0.6\Big)\Big\}, (\tau_2) = \Big\{\Big(\tfrac{v_1}{\langle[0.2,0.5]\rangle},0.5\Big),\Big(\tfrac{v_2}{\langle[0.6,0.9]\rangle},0.7\Big)\Big\}, (\tau_3) = \Big\{\Big(\tfrac{v_1}{\langle[0.3,0.8]\rangle},0.7\Big),\Big(\tfrac{v_2}{\langle[0.8,0.8]\rangle},0.4\Big)\Big\}\Big\}.$$

Step 3: Compute the ratio similarity between F_η and $\mathcal{G}_\varphi$ according to Definition 13 given above.

$$\ddot{M}(\eta(\tau_1),\varphi(\tau_1)) = 1 - \frac{\sum_{j=1}^{2} |\eta_1(\tau_1)-\varphi_1(\tau_1)|}{\sum_{j=1}^{2} |\eta_1(\tau_1)+\varphi_1(\tau_1)|}$$

$$=1 - \frac{|1-0.2|+|1-0.5|}{|1+0.2|+|1+0.5|} = 0.14$$

Similarly we get, $\ddot{M}(\eta(\tau_2),\varphi(\tau_2)) = 0.25$, $\ddot{M}(\eta(\tau_3),\varphi(\tau_3)) = 0.29$
and

$$\ddot{M}_1(\mathcal{F}(\tau_1),\mathcal{G}(\tau_1)) = 1 - \frac{1}{\sqrt{n}}\sqrt{\sum_{i=1}^{n}\Big(\dot{\phi}_{\mathcal{F}(\tau_1)}(v_j) - \dot{\phi}_{\mathcal{G}(\tau_1)}(v_j)\Big)^2},$$

$$= 1 - \frac{1}{\sqrt{2}}\sqrt{(1-0.45)^2+(1-0.45)^2} = 0.78$$

Similarly, we get the rest of the values in Table 2

Table 2. Valudes of $\ddot{M}_1(\mathcal{F}(\tau_i),\mathcal{G}(\tau_i))$ and $\ddot{M}(\eta(\tau_i),\varphi(\tau_i))$

$\ddot{M}_1(\mathcal{F}(\tau_i),\mathcal{G}(\tau_i))$	**Degree**	$\ddot{M}(\eta(\tau_i),\varphi(\tau_i))$	**Degree**
$\ddot{M}_1(\mathcal{F}(\tau_1),\mathcal{G}(\tau_1))$	0.23	$\ddot{M}(\eta(\tau_2),\varphi(\tau_2))$	0.14
$\ddot{M}_1(\mathcal{F}(\tau_2),\mathcal{G}(\tau_2))$	0.37	$\ddot{M}(\eta(\tau_2),\varphi(\tau_2))$	0.25
$\ddot{M}_1(\mathcal{F}(\tau_3),\mathcal{G}(\tau_3))$	0.32	$\ddot{M}(\eta(\tau_3),\varphi(\tau_3))$	0.29
$\ddot{M}(\mathcal{F}(\tau),\mathcal{G}(\tau)) = 0.37$		$\ddot{M}(\eta(\tau),\varphi(\tau)) = 0.29.$	

Table 2 describes the interaction process between each of the values $\ddot{M}_1(\mathcal{F}(\tau_i),\mathcal{G}(\tau_i))$ and $\ddot{M}(\eta(\tau_i),\varphi(\tau_i))$ where the greatest value for each of $\ddot{M}_1(\mathcal{F}(\tau_i),\mathcal{G}(\tau_i))$ and $\ddot{M}(\eta(\tau_i),\varphi(\tau_i))$was chosen according to the above algorithm. Then, the similarity measure between PI-FSSs $\mathcal{F}_\eta$ for illness and $\mathcal{G}_\varphi$ for the patient person compute as following: $\hat{S}\left(\mathcal{F}_\eta,\mathcal{G}_\varphi\right) = 0.37\times 0.29 = 0.10$ (The patient has no dengue fever).

7. CONCLUSION

In this work, we succeeded in studying an innovative fuzzy model called the possibility of an interval-valued fuzzy soft set (PI-VFSS) in order to improve the performance of previous fuzzy concepts when we assign an evaluation score or probability to IV-FSS. This article has been divided into two parts. In the first part, the possibility of an interval-valued fuzzy soft set (PI-VFSS) is improved. Some properties and some essential set theory were set up on PIV-FSS (these properties emerge from the laws and properties of set theory, to which fuzzy theory belongs, and its extension), as well as some numerical examples to illustrate these technicals. Also, using this method, we proposed an algorithm to solve the assumed problem in the decision-making problem. In the second part, we succeeded in applying similarity measures to this method by computing the similarity ratio between PI-VFSSs. Then, these measures are applied to the medical diagnosis to discover if the patient has dengue fever or not. Finally, for further work on these topics, we recommend developing these tools by integrating them with some other mathematical structures, such as algebraic structures [36]-[37], topological structures [38]-[39], and other ideas[40]-[43]. these properties emerge from the laws and properties of set theory to which the fuzzy theory belongs and its extension.

FUNDING

None

ACKNOWLEDGEMENT

The first author would like to thank the reviewers for providing useful suggestions, allowing for the improved presentation of this paper.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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