Vertex and region colourings of planar idempotent divisor graphs of commutative rings.

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ABSTRACT: The idempotent divisor graph of a commutative ring R is a graph with vertices set in $R^* = R - \{0\}$, and any distinct vertices x and y are adjacent if and only if $x.y = e$. For some non-unit idempotent element $e^2 = e \in R$, it is denoted by $\Pi(R)$. The purpose of this work is to use some properties of ring theory and graph theory to determine the clique number, the chromatic number and the region chromatic number for each planar idempotent divisor graph of the commutative rings. furthermore, we show that the clique number is equal to the chromatic number for any planar idempotent divisor graph. Results indicate that when F_q and F_p^α are fields of orders q and p^α , respectively, where $q=2$ or 3 , p is a prime number and is a positive integer. If ring $R \cong F_q \times F_p^\alpha$, then $\chi(\Pi(R)) = \omega(\Pi(R)) = \chi^*(\Pi(R)) = 3$.

Keywords: Zero divisor graph; idempotent elements; planar graph and idempotent divisor graph.

1. INTRODUCTION

In this work, R is a finite commutative ring with an identity of $1 \neq 0$, and $Z(R)$ is a set of all-zero divisors. In 1988, Beck [1] introduced the relationship between ring and graph theories in the study of the colouring of commutative ring. Later, in 1999, Anderson and Livingston [2], modified the model by studying the zero divisor graph $G(R)$ with vertices $Z(R)^* = Z(R) - \{0\}$. For $x_1, x_2 \in Z(R)^*$ with $x_1 \neq x_2$, $\{x_1, x_2\}$ is an edge in $\Gamma(R)$ if and only if $x_1.x_2=0$. Many authors have since studied this notion (e.g. [3–6]). Moreover, there are many other definitions that connect the two theories of graph and ring (e.g., [7–9]). In 2021, Mohammad and Shuker [10] defined the idempotent divisor graph of commutative ring R with identity $1 \neq 0$. This graph is denoted by $\Pi(R)$, which has vertices set in $R^* = R - \{0\}$. For any distinct vertices x and y , x is adjacent to y if and only if $x.y = e$ for some non-unit idempotent element $e^2 = e$ in R . If R is a local ring, then the only non-unit idempotent element in R is zero. They previously assumed that $V(\Pi(R)) = Z(R)^*$, which means that $\Pi(R) = G(R)$. Furthermore, they presented some fundamental properties of this graph and its connection with $G(R)$, and they also studied the planarity of this graph [10]. In 2021 [11], we studied the size, center and distance between vertices of the idempotent divisor graph of a direct product of two fields, and we determined the general form of this graph when any one of those two fields is a field of order two.

In graph theory, for the connected simple graph G , let H be a graph. GUH is a graph with $V(GUH) = V(G) \cup V(H)$ and $E(GUH) = E(G) \cup E(H)$. $G+H$ is a graph with $V(GUH) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$. Furthermore, let u and v be non-adjacent vertices in graph G . Then, $G+\{\{u, v\}\}$ is a graph with $V(G+\{\{u, v\}\}) = V(G)$ and $E(G+\{\{u, v\}\}) = E(G) \cup \{u, v\}$. A graph is considered to be complete if each of its two vertices are adjacent. A complete graph of order n is denoted by K_n . K_1 is a graph with only one vertex. A path graph of order n denoted by P_n is a graph with $V(P_n) = \{v_i : i=1, 2, \dots, n\}$ and $E(P_n) = \{\{v_j, v_{j+1}\} : j=1, 2, \dots, n-1\}$. A cycle graph of order n denoted by C_n is a graph with $V(C_n) = \{v_i : i=1, 2, \dots, n\}$ and $E(C_n) = \{\{v_j, v_{j+1}\} : j=1, 2, \dots, n-1\} \cup \{(v_1, v_n)\}$. A graph G is b -partite, i.e. $b \in \mathbb{Z}^+$, if

it is possible to partition $V(G)$ into b partite subsets $V_1, V_2, \dots, V_b$ such that each element of $E(G)$ joins a vertex of V_i to a vertex of V_j with $i \neq j$. The task of colouring a graph G involves assigning colours to the vertices of G , with one colour assigned to each vertex. In this manner, adjacent vertices are assigned with different colours. A colouring in which k colours are used is called k -colouring. The chromatic number of G denoted by $\chi(G)$ is defined by the minimum integer k for which a graph G is k -colourable. The clique number $\omega(G)$ of a graph G is the maximum order of each complete subgraphs of G . Clearly, for any graph G , $\chi(G) \geq \omega(G)$. More precisely, $\chi(G)$ equals $\omega(G)$ for some special cases of graphs. For example, if G is a non-empty bipartite graph, then $\chi(G) = \omega(G) = 2$. If a graph is embedded in the plane, then it is called a planar graph. Not all planar graph G satisfies $\chi(G) = \omega(G)$. For example, $\chi(C_5) = 3$, but $\omega(C_5) = 2$. A planar graph G is said to be an n -region-colourable graph if the regions of G can be coloured with n or fewer colours such that adjacent regions are coloured differently. The region chromatic number $\chi^*(G)$ of a planar graph G is the minimum n for which G is n -region colourable. For the details, see [12].

In ring theory, we let R be a finite commutative ring with an identity. R is considered to be local if it contains a single maximal ideal. Ring R is a direct product of the rings R_i for $i=1, 2, \dots, n$. If $R \cong R_1 \times R_2 \times \dots \times R_n = \{(r_1, r_2, \dots, r_n) : r_i \in R_i \text{ or if } R \text{ is a non-local, then } R \text{ is a direct product of the local rings. } F_q \text{ denotes a field of order } q, \text{ where } q \text{ is a power of prime number. For the details, see [13] and [14].}$

In this study, we use ring theory and graph theory to determine the clique number, the chromatic number and the region chromatic number for each planar idempotent divisor graph of commutative rings. In Section 2, we discuss the chromatic and clique numbers of all planar idempotent divisor graphs of commutative rings, and we prove these two numbers are equal, i.e. $\chi(\Pi(R)) = \omega(\Pi(R))$ for any ring R with planar graph $\Pi(R)$. In Section 3, we present the general form of $\Pi(F_3 \times F_{p^\alpha})$ for any prime p and any positive integer α . Moreover, we elaborate the region chromatic numbers for planar idempotent divisor graphs of the commutative rings. Finally, in the last section, we present a table for the values of $\omega(\Pi(R))$, $\chi(\Pi(R))$ and $\chi^*(\Pi(R))$ for any commutative ring that has a planar idempotent divisor graph.

2. VERTEX COLOURING

In this section, we show that the clique number is equal to the chromatic number for any ring with a planar idempotent divisor graph.

2.1 CLIQUE NUMBERS AND CHROMATIC NUMBERS OF PLANAR IDEMPOTENT DIVISOR GRAPHS OF LOCAL RINGS

We begin by presenting the lemma presented in [10].

Lemma 2.1.1. *For any local ring R , a graph $\Pi(R)$ is planar if and only if R is isomorphic to one of the following rings listed in Table 2.1:*

Table 1. Planar idempotent divisor graphs of local rings.

Ring(s) type	Graph
Z_4 or $F_2[Y]/(Y^2)$	K_1
Z_9 or $F_3[Y]/(Y^2)$	K_2
$F_2[Y_1, Y_2]/(Y_1, Y_2)^2$, $Z_4[Y]/(2, Y)^2$, $F_4[Y]/(Y^2)$ or $Z_4[Y]/(Y^2 + Y + 1)$	K_3
Z_8 , $Z_4[Y]/(2Y, Y^2 - 2)$ or $F_2[Y]/(Y^3)$	$K_{1,2}$
Z_{25} or $F_5[Y]/(Y^2)$	K_4
Z_{27} , $F_3[Y]/(Y^3)$ or $Z_9[Y]/(3Y, Y^2 - 3)$.	$K_2 + 6K_1$
Z_{16} , $F_2[Y]/(Y^4)$, $Z_4[Y]/(2Y, 2Y^2, Y^3 - 2)$, $Z_4[Y]/(Y^2 - 2Y - 2)$, or $Z_4[Y]/(Y^2 - 2)$.	$K_1 + (4K_1 \cup K_2)$
$Z_8[Y]/(2Y, Y^2 - 4)$, $Z_4[Y]/(Y^2 - 2Y)$, $F_2[Y_1, Y_2]/(Y_1^2, Y_2^2 - Y_1Y_2)$ or $Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2 - Y_1Y_2 - 2, 2Y_1, 2Y_2)$.	$K_1 + (K_2 \cup C_4)$
$Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2, Y_1Y_2 - 2, 2Y_1, 2Y_2)$, $Z_4[Y]/(Y^2)$ or $F_2[Y_1, Y_2]/(Y_1^2, Y_2^2)$.	$K_1 + 3K_2$

Then, we determine the clique numbers and chromatic numbers of the idempotent divisor graphs of the rings in lemma 2.1.1. Clearly, $\chi(K_i) = i = \omega(K_i)$ for $i=1, 2, 3, 4$. Furthermore, $\chi(K_{1,2}) = 2 = \omega(K_{1,2})$.

The graphs $K_2 + 6K_1$, $K_1 + (4K_1 \cup K_2)$, $K_1 + (K_2 \cup C_4)$ and $K_1 + 3K_2$ have the following forms:

In the figures, $\chi(K_2 + 6K_1) = 3 = \omega(K_2 + 6K_1)$, $\chi(K_1 + (4K_1 \cup K_2)) = 3 = \omega(K_1 + (4K_1 \cup K_2))$, $\chi(K_1 + (K_2 \cup C_4)) = 3 = \omega(K_1 + (K_2 \cup C_4))$ and $\chi(K_1 + 3K_2) = 3 = \omega(K_1 + (K_1 + 3K_2))$.

Proposition 2.1.2. *If R is a local ring and $\Pi(R)$ is a planar graph, then $\chi(\Pi(R)) = \omega(\Pi(R))$.*

FIGURE 1. (a) $K_1 + (4K_1 \cup K_2)$ (b) $K_2 + 6K_1$

FIGURE 2. (a) $K_1 + (K_2 \cup C_4)$ (b) $K_1 + 3K_2$

2.2 CLIQUE NUMBERS AND CHROMATIC NUMBERS OF PLANAR IDEMPOTENT DIVISOR GRAPHS OF NON-LOCAL RINGS

We begin this section by presenting the following lemmas:

Lemma 2.2.1. [10]: Let R be a non-local ring. The graph $\Pi(R)$ is planar if and only if R is isomorphic to one of the rings $F_2 \times F_q$, $F_3 \times F_q$, $F_2 \times Z_4$, $F_2 \times F_2[Y]/(Y^2)$, $F_2 \times Z_9$ or $F_2 \times F_3[Y]/(Y^2)$, where q is a power of prime.

Lemma 2.2.2. [10] Let $R \cong K_1 \times K_2$, where K_1 and K_2 are fields. Then, $(\Pi(R)) = 3$.

Firstly, we give the form of the idempotent divisor graphs of $F_2 \times Z_4$, $F_2 \times F_2[Y]/(Y^2)$, $F_2 \times Z_9$ and $F_2 \times F_3[Y]/(Y^2)$. Clearly, $\Pi(F_2 \times Z_4) \cong \Pi(F_2 \times F_2[Y]/(Y^2)) \cong K_1 + 3K_2$. Furthermore, $\Pi(F_2 \times Z_9) \cong \Pi(F_2 \times F_3[Y]/(Y^2))$ has the following form:

FIGURE 3. $\Pi(F_2 \times Z_9) \cong \Pi(F_2 \times F_3[Y]/(Y^2))$

As shown in the figure, $\chi(\Pi(F_2 \times Z_4)) = \omega(\Pi(F_2 \times Z_4)) = 3$, and $(\Pi(F_2 \times Z_9)) = \omega(\Pi(F_2 \times Z_9)) = 4$.

To find $\chi(\Pi(F_2 \times F_{p^\alpha}))$ for any prime number p and any positive integer α , we need to derive the following two results, similar to those in [11], for our subsequent results.

Lemma 2.2.3. [11] $\Pi(F_2 \times F_{p^\alpha}) \cong K_1 + (2K_2 \cup (\bigcup_{i=1}^{\frac{p^\alpha-3}{2}} P_4))$, where p is a prime number greater than 2, $\alpha \in \mathbb{Z}^+$.

Lemma 2.2.4. [11] $\Pi(F_2 \times F_{p^\alpha}) \cong K_1 + (K_2 \cup (\bigcup_{i=1}^{p^\alpha-1} P_4))$, where $p=2$, $\alpha \in \mathbb{Z}^+$.

Theorem 2.2.5. $\chi(\Pi(F_2 \times F_{p^\alpha})) = \omega(\Pi(F_2 \times F_{p^\alpha})) = 3$ for any prime p and any positive integer α .

Proof. According to **Lemma 2.2.2**, we have $\omega(\Pi(F_2 \times F_{p^\alpha})) = 3$. If $p^\alpha = 2$ or 3 , then $\Pi(F_2 \times F_2) \cong K_3$ and $\Pi(F_2 \times F_3) \cong K_1 + 2K_2$. Therefore, $\chi(\Pi(F_2 \times F_2)) = \chi(\Pi(F_2 \times F_3)) = 3$. Subsequently, let $p^\alpha \geq 4$.

We have the following two cases:

Case 1: If p is a prime number greater than 2, then from the proof of **Lemma 2.2.3** [11], we have $V(\Pi(F_2 \times F_{p^\alpha})) = F_2 \times F_{p^\alpha} - \{0\} = \bigcup_{i=1}^4 A_i$, where $A_1 = \{(1,0)\}$, $A_2 = \{(0,1), (1,1)\}$, $A_3 = \{(0,-1), (1,-1)\}$ and $A_4 = \bigcup_{i=1}^{\frac{p^\alpha-3}{2}} B_i$ such that $B_i = \{(0, f_i), (0, f_i^{-1}), (1, f_i), (1, f_i^{-1}) : f_i \in F_{p^\alpha} - \{0, 1, -1\}\}$ for $i=1, 2, \dots, \frac{p^\alpha-3}{2}$. Thus, $\Pi(F_2 \times F_{p^\alpha})$ is

three partite. Let $X_1 = A_1$, $X_2 = \{(0,1), (0,-1)\} \cup \left\{ \bigcup_{i=1}^{\frac{p^\alpha-3}{2}} \{(0, f_i), (1, f_i) : f_i \in F_{p^\alpha} - \{0, 1, -1, f_i^{-1}\}\} \right\}$ and $X_3 = \{(1,1), (1,-1)\} \cup (A_4 - X_2)$. Clearly, any two vertices in X_i are not adjacent for any $i=2,3$. As $\omega(\Pi(F_2 \times F_{p^\alpha})) = 3$ and $\chi(\Pi(F_2 \times F_{p^\alpha})) \geq \omega(\Pi(F_2 \times F_{p^\alpha}))$, the graph $\Pi(F_2 \times F_{p^\alpha})$ is three partite. Therefore, $\chi(\Pi(F_2 \times F_{p^\alpha})) = 3$.

Case 2: If $p=2$, from the proof of **Lemma 2.2.4** and by using the same method for the proof of case 1, then we have $V(\Pi(F_2 \times F_{p^\alpha})) = F_2 \times F_{p^\alpha} - \{0\} = \bigcup_{i=1}^3 A_i$, where $A_1 = \{(1,0)\}$, $A_2 = \{(0,1), (1,1)\}$ and $A_3 = \bigcup_{i=1}^{\frac{p^\alpha-2}{2}} B_i$ such that $B_i = \{(0, f_i), (0, f_i^{-1}), (1, f_i), (1, f_i^{-1}) : f_i \in F_{p^\alpha} - \{0, 1\}\}$ for $i=1, 2, \dots, \frac{p^\alpha-2}{2}$. Let $X_1 = A_1$, $X_2 = \{(0,1)\} \cup \left\{ \bigcup_{i=1}^{\frac{p^\alpha-2}{2}} \{(0, f_i), (1, f_i) : f_i \in F_{p^\alpha} - \{0, 1, f_i^{-1}\}\} \right\}$ and $X_3 = \{(1,1)\} \cup (A_3 - X_2)$. Clearly, any two vertices in X_i are not adjacent for any $i=2,3$. Similar to the scenario in case 1, the graph $\Pi(F_2 \times F_{p^\alpha})$ is three partite. Therefore, $\chi(\Pi(F_2 \times F_{p^\alpha})) = 3$ also in case 2.

Therefore, $\omega(\Pi(F_2 \times F_{p^\alpha})) = \chi(\Pi(F_2 \times F_{p^\alpha})) = 3$ for any prime p and any positive integer α .

Example 2.2.6. The graphs $\Pi(F_2 \times F_4)$ and $\Pi(F_2 \times F_7)$ have the following forms:

FIGURE 4. (a) $\Pi(F_2 \times F_4)$

(b) $\Pi(F_2 \times F_7)$

Theorem 2.2.7. $\chi(\Pi(F_3 \times F_{p^\alpha})) = \omega(\Pi(F_3 \times F_{p^\alpha})) = 3$ for any prime number p and any positive integer α .

Proof. According to **Lemma 2.2.2**, we have $\omega(\Pi(F_3 \times F_{p^\alpha})) = 3$. If $p^\alpha = 2$, then $\Pi(F_3 \times F_2) \cong K_1 + 2K_2$. Therefore, $\chi(\Pi(F_3 \times F_2)) = 3$. If $p^\alpha = 3$, then $\Pi(F_3 \times F_3)$ has the following form:

FIGURE 5. $\Pi(F_3 \times F_3)$

Furthermore, $\chi(\Pi(F_3 \times F_3)) = 3$. Then, we let $p^\alpha \geq 4$.

We have the following two cases:

Case 1: If $p=2$, then for $a \in 2^\alpha$, $a=a^{-1}$ if and only if $a=1$. Let $R^* = F_3 \times F_{p^\alpha} - \{0\} = X \cup Y$ such that $X = \{(1,0), (-1,0), (1,1), (-1,1), (0,1)\}$ and $Y = \{(0,y), (0,y^{-1}), (1,y), (1,y^{-1}), (-1,y), (-1,y^{-1}) : y \in F_{2^\alpha} - \{0,1\}\}$. Suppose that $X_1 = \{(1,0), (-1,0)\}$, $X_2 = \{(1,1), (-1,1)\} \cup \{(0,y), (1,y), (-1,y) : y \in F_{2^\alpha} - \{0,1,y^{-1}\}\}$ and $X_3 = \{(0,1)\} \cup (Y - X_2)$. Clearly, any two vertices in X_i are not adjacent for any $i=1,2,3$. As $\omega(\Pi(F_3 \times F_{p^\alpha})) = 3$ and $\chi(\Pi(F_3 \times F_{p^\alpha})) \geq \omega(\Pi(F_3 \times F_{p^\alpha}))$, then the graph $\Pi(F_3 \times F_{p^\alpha})$ is three partite. Therefore, $\chi(\Pi(F_3 \times F_{p^\alpha})) = 3$.

Case 2: If p is a prime number greater than 2, then for $a \in p^\alpha$, $a=a^{-1}$ if and only if $a=\pm 1$. As a proof of case 1, $F_3 \times F_{p^\alpha} - \{0\} = X \cup Y$ such that $X = \{(1,0), (-1,0), (1,1), (-1,1), (0,1), (0,-1), (1,-1), (-1,-1)\}$ and $Y = \{(0,y), (0,y^{-1}), (1,y), (1,y^{-1}), (-1,y), (-1,y^{-1}) : y \in F_{2^\alpha} - \{0,1,-1\}\}$. Let $X_1 = \{(1,0), (-1,0)\}$, $X_2 = \{(1,1), (-1,1), (1,-1), (-1,-1)\} \cup \{(0,y), (1,y), (-1,y) : y \in F_{2^\alpha} - \{0,1,-1,y^{-1}\}\}$ and $X_3 = \{(0,1), (0,-1)\} \cup (Y - X_2)$. Clearly, any two vertices in X_i are not adjacent for any $i=1,2,3$. The graph $\Pi(F_3 \times F_{p^\alpha})$ is three partite. Therefore, $\chi(\Pi(F_3 \times F_{p^\alpha})) = 3$ in this case.

Therefore, $\chi(\Pi(F_3 \times F_{p^\alpha})) = \omega(\Pi(F_3 \times F_{p^\alpha})) = 3$ for any prime p and any positive integer α .

Example 2.2.8. The graphs $\Pi(F_3 \times F_4)$ and $\Pi(F_3 \times F_5)$ have the following forms:

FIGURE 6. (a) $\Pi(F_3 \times F_4)$

(b) $\Pi(F_3 \times F_5)$

On the basis of **Proposition 2.1.2** and **Theorems 2.2.5** and **2.2.7**, we have the following main result.

Theorem 2.2.9. Let R be any ring with a planar idempotent divisor graph. Then, $\chi(\Pi(R)) = \omega(\Pi(R))$.

3. REGION COLOURING

In this section, we determine the region chromatic number for planar idempotent divisor graphs of commutative rings.

3.1 REGION CHROMATIC NUMBERS FOR PLANAR IDEMPOTENT DIVISOR GRAPHS OF LOCAL RINGS

According to Lemma 2.1.1, we have $\chi^*(K_i) = \chi^*(K_{1,2}) = 1$ for $i=1,2$, $\chi^*(K_3) = 2$ and $\chi^*(K_4) = 4$. We can colour the regions of all idempotent divisor graphs of local rings as the follows:

As shown in the figure, $\chi^*(K_1 + (4K_1 \cup K_2)) = \chi^*(K_1 + 3K_2) = 2$ and $\chi^*(K_2 + 6K_1) = \chi^*(K_1 + (K_2 \cup C_4)) = 3$.

3.2 REGION CHROMATIC NUMBERS FOR PLANAR IDEMPOTENT DIVISOR GRAPHS OF NON-LOCAL RINGS

According to Lemma 2.2.1, the idempotent divisor graph of the non-local ring R is planar if and only if $R \cong F_2 \times Z_4$, $F_2 \times F_2[Y]/(Y^2)$, $F_2 \times Z_9$, $F_2 \times F_3[Y]/(Y^2)$, $F_2 \times F_{p^\alpha}$ or $F_3 \times F_{p^\alpha}$ for any prime number p and any positive integer α . Clearly, $\Pi(F_2 \times Z_4) \cong \Pi(F_2 \times F_2[Y]/(Y^2)) \cong K_1 + 3K_2$. Hence, $\chi^*(\Pi(F_2 \times Z_4)) = 2$.

The region colouring of $\Pi(F_2 \times Z_9)$ and $\Pi(F_2 \times F_3[Y]/(Y^2))$ is as follows:

Clearly, $\chi^*(\Pi(F_2 \times Z_9)) = 3$. Subsequently, we find the region chromatic number for $\Pi(F_2 \times F_{p^\alpha})$. If $p^\alpha = 2$ or 3 , then $\Pi(F_2 \times F_2) \cong K_3$ and $\Pi(F_2 \times F_3) \cong K_1 + 2K_2$. Therefore $\chi^*(\Pi(F_2 \times F_2)) = \chi^*(\Pi(F_2 \times F_3)) = 2$. Similarly, we find $\chi^*(\Pi(F_2 \times F_{p^\alpha}))$ for $p^\alpha \geq 4$.

FIGURE 7. (a) $K_1 + (4K_1 \cup K_2)$ (b) $K_2 + 6K_1$

FIGURE 8. (a) $K_1 + (K_2 \cup C_4)$ (b) $K_1 + 3K_2$

FIGURE 9. $\Pi(F_2 \times Z_9) \cong \Pi(F_2 \times F_3[Y] / (Y^2)$

Theorem 3.2.1. $\chi^*(\Pi(F_2 \times F_{p^\alpha}))=3$ for any prime number p and any positive integer α with $p^\alpha \geq 4$.

Proof. We have the following two cases:

Case 1: If p is a prime number greater than 2, according to **Lemma 2.2.3** we have $\Pi(F_2 \times F_{p^\alpha}) \cong K_1 + (2K_2 \cup (\bigcup_{i=1}^{\frac{p^\alpha-3}{2}} P_4))$. Clearly, $\chi^*(K_1 + P_4)=3$ and $\chi^*(K_1 + \bigcup_{i=1}^{\frac{p^\alpha-3}{2}} P_4)=3$. Hence, $\chi^*(2K_2 \cup (\bigcup_{i=1}^{\frac{p^\alpha-3}{2}} P_4))=3$.

Case 2: If $p=2$, by similarly using the same method of case 1, we have $\chi^*(\Pi(F_2 \times F_{p^\alpha}))=3$ in this case. Therefore, $\chi^*(\Pi(F_2 \times F_{p^\alpha}))=3$.

Example 3.2.2. The colours of the regions of the graph $\Pi(F_2 \times F_4)$ and $\Pi(F_2 \times F_7)$ are as follows:

FIGURE 10. (a) $\Pi(F_2 \times F_4)$

(b) $\Pi(F_2 \times F_7)$

For the graph $\Pi(F_3 \times F_{p^\alpha})$, where p is a prime number and α is a positive integer. If $p^\alpha=2$, then $\Pi(F_3 \times F_{p^\alpha}) = \Pi(F_3 \times F_2) \cong K_1 + 2K_2$. Therefore, $\chi^*(\Pi(F_3 \times F_2))=2$. If $p^\alpha=3$, then the colour of the regions of the graph $\Pi(F_3 \times F_3)$ is as follows:

FIGURE 11. $\Pi(F_3 \times F_3)$

Therefore, $\chi^*(\Pi(F_3 \times F_3))=2$.

Theorem 3.2.3. $\chi^*(\Pi(F_3 \times F_{p^\alpha}))=3$ for any prime number p and any positive integer α with $p^\alpha \geq 4$.

Proof. We have the following two cases:

Case 1: If $p=2$, according to **Theorem 2.2.7**, we have $F_3 \times F_{p^\alpha} - \{0\} = X \cup Y$ such that $X = \{(1,0), (-1,0), (1,1), (-1,1), (0,1)\}$ and $Y = \{(0,y), (0,y^{-1}), (1,y), (1,y^{-1}), (-1,y), (-1,y^{-1}) : y \in F_{2^\alpha} - \{0,1\}\}$. Clearly, $|Y| = 3(2^\alpha) - 6$. Then, let $F_{2^\alpha}^* = \{a_i \in F_{2^\alpha} : a_i \neq 0, 1 \text{ nor } a_i^{-1} \text{ for } i=1, 2, \dots, \frac{2^\alpha-2}{2}\}$, and let $Y = \bigcup_{i=1}^{\frac{3(2^\alpha)-6}{6}} A_i = \bigcup_{i=1}^{2^{\alpha-1}-1} A_i$ such that $A_i = \{(0, a_i), (0, a_i^{-1}), (1, a_i), (1, a_i^{-1}), (-1, a_i), (-1, a_i^{-1}) : a_i \in F_{2^\alpha}^*\}$. Clearly, $-K_1 + 2K_2$. For any $i=1, 2, \dots, 2^{\alpha-1}-1$, we have the following form:

The vertex $(1,0)$ is adjacent to the vertices $(1, a_i), (1, a_i^{-1}), (0, a_i)$ and $(0, a_i^{-1})$ of $-$ but not to $(-1, a_i), (-1, a_i^{-1})$ for any $i=1, 2, \dots, 2^{\alpha-1}-1$ in the graph $\Pi(F_3 \times F_{p^\alpha})$. Similarly, the vertex $(-1,0)$ is adjacent only to $(-1, a_i), (-1, a_i^{-1}), (0, a_i)$ and $(0, a_i^{-1})$ of $-$ in the graph $\Pi(F_3 \times F_{p^\alpha})$; otherwise, there is no vertex in X adjacent to any vertex in $-$ for $i=1, 2, \dots, 2^{\alpha-1}-1$ in the graph $\Pi(F_3 \times F_{p^\alpha})$. Then, we can express the general form of $\Pi(F_3 \times F_{2^\alpha})$ as follows:

$\Pi(F_3 \times F_{2\alpha}) = (-) + E$, where E is the following subset of $E(\Pi(F_3 \times F_{p^\alpha}))$:

$E = \{ \{ (j,0), (k,x) \} : j=1,-1, k=0, x=a_i, a_i^{-1} \text{ and } i=1,2,\dots, 2^{\alpha-1}-1 \}$.

If $\alpha=2$, then $F_{2\alpha}^* = \{a_1\}$. Hence, $Y=A_1$ such that $A_1 = \{ (0,a_1), (0,a_1^{-1}), (1,a_1), (1,a_1^{-1}), (-1,a_1), (-1,a_1^{-1}) \}$. Thus, $F_3 \times F_{p^\alpha} = X \cup A_1$. Furthermore, $\Pi(F_3 \times F_{2\alpha}) = \Pi(F_3 \times F_4) = (-) + \{ \{ (1,0), (0,a_1) \}, \{ (1,0), (0,a_1^{-1}) \}, \{ (1,0), (1,a_1) \}, \{ (1,0), (1,a_1^{-1}) \}, \{ (-1,0), (0,a_1) \}, \{ (-1,0), (0,a_1^{-1}) \}, \{ (-1,0), (-1,a_1) \}, \{ (-1,0), (-1,a_1^{-1}) \} \}$. The corresponding forms are as follows:

$\Pi(F_3 \times F_{22})$ has the following colours for the regions:

As shown in the figure, $\chi^*(\Pi(F_2 \times F_4)) = 3$.

Let $\alpha=3$; then, $F_{2\alpha}^*=\{a_1, a_2, a_3\}$. Hence, $Y=A_1 \cup A_2 \cup A_3$ such that $A_i=\{(0, a_i), (0, a_i^{-1}), (1, a_i), (1, a_i^{-1}), (-1, a_i), (-1, a_i^{-1})\}$ for $i=1, 2, 3$. Thus, $F_3 \times F_{p\alpha} = X \cup A_1 \cup A_2 \cup A_3$, Moreover, $\Pi(F_3 \times F_{2\alpha}) = \Pi(F_3 \times F_8) = (-\{(j, 0), (k, x)\} : j=1, -1, k=0, j, x=a_i, a_i^{-1}, \text{ in which } i=1, 2, 3\}$ has the following form:

FIGURE 15. $\Pi(F_3 \times F_8)$

$\Pi(F_3 \times F_{2\alpha})$ has the following colours of the regions:

FIGURE 16. Region colouring of $\Pi(F_3 \times F_8)$

The figures also show that $\chi^*(\Pi(F_3 \times F_8)) = 3$.

Similarly, for any $\alpha > 3$, we have $\chi^*(\Pi(F_3 \times F_{2\alpha})) = 3$.

Case 2: If p is a prime number with $p \neq 2$, by using the same method as a proof of case 1, then $F_3 \times F_{p\alpha} - \{0\} = X \cup Y$.

$X = \{(1, 0), (-1, 0), (1, 1), (-1, 1), (0, 1), (0, -1), (1, -1), (-1, -1)\}$ and $Y = \bigcup_{i=1}^{\frac{3(p^\alpha-9)}{6}} B_i = \bigcup_{i=1}^{\frac{p^\alpha-3}{2}} B_i$ such that $B_i = \{(0, b_i), (0, b_i^{-1}), (1, b_i), (1, b_i^{-1}), (-1, b_i), (-1, b_i^{-1}) : b_i \in F_{p\alpha}^*\}$, where $F_{p\alpha}^* = \{b_i \in F_{2\alpha} : a_i \neq 0, 1, -1, \text{ nor } b_i^{-1} \text{ for } i=1, 2, \dots, \frac{p^\alpha-3}{2}\}$. Clearly, – for any $i=1, 2, \dots, \frac{p^\alpha-3}{2}$. Subsequently, – has the following form:

Similar to the scheme in case 1, the general form of the graph $\Pi(F_3 \times F_{p\alpha})$ is as follows:

$\Pi(F_3 \times F_{p\alpha}) = (-) + E$, where E is given by $E = \left\{ \{(j, 0), (k, x)\} : j = 1, -1, k = 0, j, x = b_i, b_i^{-1}, \text{ and } i = 1, 2, \dots, \frac{p^\alpha-3}{2} \right\}$.

Hence, $\chi^*(\Pi(F_3 \times F_{p\alpha})) = 3$ in this case. Therefore, $\chi^*(\Pi(F_3 \times F_{p\alpha})) = 3$.

Example 3.2.4. The colours of the regions of the graph $\Pi(F_3 \times F_5)$ is as follows:

FIGURE 17. $\langle X \rangle$

FIGURE 18. $\Pi(\mathbf{F}_3 \times F_5)$

4. CONCLUSION

On the basis of the discussions in the last two sections, we have obtained the following table for the clique, the chromatic and the region chromatic numbers for each planar idempotent divisor graph of the commutative rings.

Table 2. Values of $\omega(\Pi(R))$, $\chi(\Pi(R))$ and $\chi^*(\Pi(R))$ for the planar idempotent divisor graphs of commutative rings.

Ring R	Type of R	Order of $\Pi(R)$	$\chi(\Pi(R))$	$\omega(\Pi(R))$	$\chi^*(\Pi(R))$
Z_4 or $F_2[Y]/(Y^2)$	Local	1	1	1	1
Z_9 or $F_3[Y]/(Y^2)$	Local	2	2	2	1
$F_2[Y_1, Y_2]/(Y_1, Y_2)^2$, $Z_4[Y]/(2, Y)^2$, $F_4[Y]/(Y^2)$ or $Z_4[Y]/(Y^2 + Y + 1)$	Local	3	3	3	2
Z_8 , $Z_4[Y]/(2Y, Y^2 - 2)$ or $F_2[Y]/(Y^3)$	Local	3	2	2	1
Z_{25} or $F_5[Y]/(Y^2)$	Local	4	4	4	4
Z_{16} , $F_2[Y]/(Y^4)$, $Z_4[Y]/(2Y, 2Y^2, Y^3 - 2)$, $Z_4[Y]/(Y^2 - 2Y - 2)$, or $Z_4[Y]/(Y^2 - 2)$.	Local	7	3	3	2
$Z_8[Y]/(2Y, Y^2 - 4)$, $Z_4[Y]/(Y^2 - 2Y)$, $F_2[Y_1, Y_2]/(Y_1^2, Y_2^2 - Y_1Y_2)$ or $Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2 - Y_1Y_2 - 2, 2Y_1, 2Y_2)$.	Local	7	3	3	3
Z_{27} , $F_3[Y]/(Y^3)$ or $Z_9[Y]/(3Y, Y^2 - 3)$.	Local	8	3	3	3
$Z_4[Y_1, Y_2]/(Y_1^2, Y_2^2, Y_1Y_2 - 2, 2Y_1, 2Y_2)$, $Z_4[Y]/(Y^2)$ or $F_2[Y_1, Y_2]/(Y_1^2, Y_2^2)$.	Local	7	3	3	2
$F_2 \times Z_4$ or $F_2 \times F_2[Y]/(Y^2)$	Non-local	7	3	3	2
$F_2 \times Z_9$ or $F_2 \times F_3[Y]/(Y^2)$	Non-local	17	4	4	3
$F_2 \times F_2$	Non-local	3	3	3	2
$F_2 \times F_3$	Non-local	5	3	3	2
$F_3 \times F_3$	Non-local	8	3	3	2
$F_2 \times F_{p^\alpha}$, for any prime number p and any positive integer α with $p^\alpha \geq 4$.	Non-local	$2p^\alpha - 1$	3	3	3
$F_3 \times F_{p^\alpha}$, for any prime number p and any positive integer α with $p^\alpha \geq 4$.	Non-local	$3p^\alpha - 1$	3	3	3

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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