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Numerical Approach for Solving fuzzy Integro-Differential Equations

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ABSTRACT: In this paper, we consider a new class of fuzzy functions called Fuzzy Integro- Differential Equations. Some numerical methods, such as Euler, have been used to determine the solutions of these equations. We extend these numerical techniques to find the optimal solutions by using control parameters, the extended difference Euler technique is used for this. Based on the parametric form of the fuzzy number, the Integro- Differential Equation is divided into two systems of the second kind. Illustrative examples are given to demonstrate the high precision and good performance of the new class. Graphical representations reveal the symmetry between lower and upper-cut represent of fuzzy solutions and may be helpful for a better understanding of fuzzy models in artificial intelligence and medical science. The results show that the extended Euler method is more accurate in terms of absolute error.

Keywords: Fuzzy integro-differential equations, extended difference Euler method, Exact solution, Approximate solution, fuzzy parameter, control parameters.

1. INTRODUCTION

Fuzzy integro-differential equations and their solutions are one of the key findings in the field of fuzzy theory. [1] Speech processing, biological signal processing, science, electroencephalogram classification EEG, economics, feminism, and communication systems are just a few of the numerous domains that rely on these equations to model dynamic systems [2–6]. As a matter of fact, most situations in nature are fuzzy and unclear, making the models' rules all the more crucial.

Since Zadeh in 1972 [7], both types fuzzy differential equations and integro differential equations have been studied extensively [8]. Fuzzy derivative and its generalizations was introduced by Seikkala [9–11]. On the other hand, the fuzzy integral was introduced by Dubois and Prade [12], they showed that fuzzy differential equation in the following form

$$\begin{cases} y'(t,r) = g(t, y(t,r)) \\ y(t_0, r) = y_0 \end{cases} \tag{1}$$

has a unique solution in fuzzy case under the condition g satisfy the Lipschitz. Fuzzy Cauchy problem was studied by Kaleva [5]. Hajighasemi et.al. [10] investigated existence and uniqueness of solutions for fuzzy integrodifferntial equations with fuzzy kernel function. Ishak and Chaini [8] proposed the new numerical method based trapezoidal technique to solve first order fuzzy problem. There has been a new intrusion into the realm of fuzzy mathematics with the study of fuzzy integro differential equations. The analytical methods for determining the exact solutions of fuzzy integro differential equations are highly challenging, thus the best approach to resort to it is through the use of a numerical technique.

We create an effective approach for computing the approximate solutions of the suggested model, and we find several features that connect fuzzy theory and integro-differential equations. Furthermore, we demonstrate that the control parameters significantly contribute to the approximation of solutions for fuzzy equations.

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This paper follows the following structure: The contains and Preliminaries in Section 2. Methodologies are described in Section 3 for resolving fuzzy integro-differential equations. In Section 4, we see two cases in point. The paper's conclusion is presented in Section 5.

2. PRELIMINARIES

In this paper, we use the following notations: $X(t_n)$ and X_n are exact solution and approximate solution respectively in time t_n.

Definition 2.1. *[4]: A fuzzy number v is a fuzzy subset of a real line which it satisfies the following conditions Convexity, normality and upper semi continuous membership of bounded support.*

Any fuzzy number v can be represent by the following parametric forms $\left(\underline{v}(r), \overline{v}(r)\right), 0 \leq r \leq 1$. *That satisfies*

a) $\underline{v}(r)$ *is non-decreasing and bounded left over* $0 \leq r \leq 1$

b) $\overline{v}(r)$ *is a bounded left continuous and non-increasing over* $0 \leq r \leq 1$

For each $r \in (0,1]$ *then* $\underline{v}(r) \leq \overline{v}(r)$.

Definition 2.2. *[3]: The r-level set is defend as* $(u]^r = (s; u(s) \geq r\}, \quad 0 \leq r \leq 1$

Consequently, $(u]^r$ *can be written as close interval*

$$(u]^r = \left(\underline{u}(r), \overline{u}(r)\right]$$

Definition 2.3. *[1]: A triangular fuzzy number is a fuzzy set V in X that is characterized by a tri-ordered* (a_l, a_c, a_r) *in the space* R^3 *with* $a_l \leq a_c \leq a_r$ *such that* $(V]^0 = (a_l, a_r]$ *and* $(V]^1 = (a_c\}$. *The r-level set of a triangular fuzzy number V is given by* $(V]^r = (a_c - (1-r)(a_c - a_l), a_c + (1-r)(a_r - a_c)]$.

Proposition 2.4. *[7]: Let* $\varphi : (a,b] \times [0,1] \to X$ *be a fuzzy function such that* $\varphi(t,r) = \left(\underline{\varphi}(t,r), \overline{\varphi}(t,r)\right)$, *then, If* φ *is differentiable then* $\underline{\varphi}(t,r)$ *and* $\overline{\varphi}(t,r)$ *are differentiable functions and* $\varphi'(t,r) = \left(\underline{\varphi}'(t,r), \overline{\varphi}'(t,r)\right)$

Definition 2.5. *[2]: Let* $\varphi : [a,b] \to X$. *Then for any partition* $P = (a = t_0, t_1, t_2, \ldots, t_m = b\}$ *and* $\xi_i \in [t_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, m$ *the definite integral of* φ *over* $a, b]$ *is*

$$\int_a^b \varphi(t)\,dt = \lim_{\vartheta \to 0} M_P$$

Where, $\vartheta = max((t_{i+1} - t_i|, \ i = 0, 1, 2, \ldots, m\}$ and $M_P = \sum_{i=1}^m \varphi(\xi_i)(t_{i+1} - t_i)$

In the case φ is a fuzzy and continuous function then for each fuzzy parameter $0 \leq r \leq 1$, its definite integral exists and also [3]

$$\begin{cases} \underline{\left(\int_a^b \varphi(t,r)dt\right)} = \int_a^b \underline{\varphi}(t,r)dt \\ \overline{\left(\int_a^b \varphi(t,r)dt\right)} = \int_a^b \overline{\varphi}(t,r)dt \end{cases} \tag{2}$$

Definition 2.6. *[9]: Let* $x = \left(\underline{x}(r), \overline{x}(r)\right)$ *and* $y = \left(\underline{y}(r), \overline{y}(r)\right), 0 \leq r \leq 1$ *be fuzzy numbers. The distance between them is defined as follows*

$$d(x,y) = \left(\int_0^1 \left(\underline{x}(r) - \underline{y}(r)\right)^2 dr + \int_0^1 (\overline{x}(r) - \overline{y}(r))^2 dr\right]^{0.5} \tag{3}$$

3. METHODOLOGY DESCRIPTION

The fuzzy integro-differential equations

$$\begin{cases} X'(t,r) + P(t,r)X(t,r) = f(t,r) + \beta \int_a^b k(t,s)X(s,r)ds \\ X(a) = X_0(r) \end{cases} \tag{4}$$

Where, $\beta > 0$, k is an arbitrary given, $X^{'}(t,r)$ is a first order derivative of the fuzzy function which defined on $(a,b]$ and is already given, r is a fuzzy parameter with values in $(0,1]$, $k(t,s)$ over $s,t \in (a,b]$ is the kernel of this equation.

In parametric form, equation (4) is represented as follows

$$\begin{cases} \underline{X'}(t,r) + \underline{P(t,r)X(t,r)} = \underline{f}(t,r) + \beta \int_a^b \underline{k(t,s)X(s,r)}ds \\ \overline{X'}(t,r) + \overline{P(t,r)X(t,r)} = \bar{f}(t,r) + \beta \int_a^b \overline{k(t,s)X(s,r)}ds \\ \underline{X}(a) = \underline{X}_0(r) \\ \bar{X}(a) = \overline{X}_0(r) \end{cases} \tag{5}$$

In addition, $\underline{P(t,r)\ X(t,r)} = \underline{P}(t,r)\underline{X}(t,r)$, $\overline{P(t,r)\ X(t,r)} = \overline{P}(t,r)\overline{X}(t,r)$, $P(t,r) = \left(\underline{P}(t,r), \overline{P}(t,r)\right)$, $\underline{k(t,s)X(s,r)} = k(t,s)\underline{X}(s,r)$, $\overline{k(t,s)X(s,r)} = k(t,s)\overline{X}(s,r)$

In this work, the extended difference Euler method is proposed by improving difference Euler method and extend one step farther to given more accurate approximate results.

The formula of Euler method is

$$\begin{cases} X_n = X_{n-1} + \gamma_1 h X'_{n-1} \\ X_{n-1} = X_n - \gamma_2 h X'_n \end{cases} \tag{6}$$

Where $\gamma_1,\ \gamma_2 > 0$ are control parameters which are effective in reducing the errors of approximation and their values can be found through simulation. Equations in (6) give the following formula

$$X_n = X_{n-1} + \frac{h}{2}\left(\gamma_1 X^{'}_{n-1} + \gamma_2 X^{'}_n\right) \tag{7}$$

To use the following notations in the equations (5)

$$\underline{\psi}\left(t, r, X(t,r), \int_a^b k(t,s)\underline{X}(s,r)ds\right) = -\underline{P(t,r)\ X(t,r)} + \underline{f}(t,r) + \beta\int_a^b \underline{k(t,s)X(s,r)}ds$$

$$\overline{\psi}\left(t, r, X(t,r), \int_a^b \overline{k(t,s)X(s,r)}ds\right) = -\overline{P(t,r)\ X(t,r)} + \overline{f}(t,r) + \beta\int_a^b \overline{k(t,s)X(s,r)}ds$$

$$\begin{cases} \underline{\psi_n} = -\underline{P_n X_n} + \underline{f_n} + \beta\int_a^b \underline{k(t_n,s)X_n}ds \\ \overline{\psi_n} = -\overline{P_n X_n} + \overline{f_n} + \beta\int_a^b \overline{k(t_n,s)X_n}ds \end{cases} \tag{8}$$

Where, $\underline{P_n\ X_n} = \underline{P(t_n,r)\ X(t_n,r)}$, $\underline{f_n} = \underline{f}(t_n,r)$. $\underline{k(t_n,s)X_n} = k(t_n,s)\ \underline{X}(t_n,r)$, $\overline{P_n\ X_n} = \overline{P(t_n,r)\ X(t_n,r)}$, $\overline{f_n} = \overline{f}(t_n,r)$ and $\overline{k(t_n,s)X_n} = k(t_n,s)\overline{X(t_n,r)}$

Now, applying these notations and the formula in (7) on equations in (5), we have

$$\begin{cases} \underline{X_n} = \underline{X_{n-1}} + \frac{h}{2}\left(\gamma_1\underline{\psi_{n-1}} + \gamma_2\underline{\psi_n}\right) \\ \overline{X_n} = \overline{X_{n-1}} + \frac{h}{2}\left(\gamma_1\overline{\psi_{n-1}} + \gamma_2\overline{\psi_n}\right) \end{cases} \tag{9}$$

Now, using composite Simpsons on with n subintervals and s belong to $(a,b]$, the integral part of equations in (8) is approximated by

$$\underline{I}_0 = \frac{2h}{3}\left(k(t_0,t_0)\underline{X_0}\right)$$

$$\overline{I}_0 = \frac{2h}{3}\left(k(t_0,t_0)\overline{X_0}\right)$$

$$\underline{I}_{1} = \frac{h}{3}\left(k(t_1,t_0)\underline{X_0} + k(t_1,t_1)\underline{X_1}\right)$$

$$\bar{I}_1 = \frac{h}{3}\left(k(t_1,t_0)\overline{X_0} + k(t_1,t_1)\overline{X_1}\right)$$

$$\begin{cases} \underline{I}_n = \int_a^b \frac{k(t_n,s)\underline{X}_n}{ds} = \frac{h}{3}\left(k(t_n,t_0)\underline{X_0} + 4\sum_{k=1}^{n-1} k(t_n,t_k)\underline{X_k} + k(t_n,t_n)\underline{X_n}\right) \\ \bar{I}_n = \int_a^b \overline{k(t_n,s)X_n} ds = \frac{h}{3}\left(k(t_n,t_0)\overline{X_0} + 4\sum_{k=1}^{n-1} k(t_n,t_k)\overline{X_k} + k(t_n,t_n)\overline{X_n}\right) \end{cases} \tag{10}$$

Consequently, the equations in (8) become

$$\begin{cases} \underline{\underline{\underline{\psi_n}}} = -\underline{\underline{\underline{P_nX_n}}} + \underline{\underline{f_n}} + \beta \underline{I}_n \\ \overline{\psi_n} = -\overline{P_nX_n} + \overline{f_n} + \beta\bar{I}_n \end{cases} \tag{11}$$

By substituting equations (8) and (10) in equations (9), we get on the following formulas n=2,3,

$$\overline{X_n} = \left(1 + \frac{h}{2}\gamma_2\,\overline{P_n} - \frac{h^2}{6}\gamma_2\beta k(t_n,t_n)\right\}^{-1}\left(\overline{X_{n-1}} + \frac{h}{2}\left(\gamma_1\overline{\psi_{n-1}}\right) + \frac{h}{2}\gamma_2\left(\overline{f_n} + \frac{\beta h}{3}\left(k(t_n,t_0)\overline{X_0} + 4\sum_{k=1}^{n-1} k(t_n,t_k)\overline{X_k}\right)\right\}\right\}$$

$$\underline{X_1} = \left(1 + \frac{h}{2}\gamma_2\,\underline{P_1} - \frac{h^2}{6}\gamma_2\beta k(t_1,t_1)\right\}^{-1}\left(\underline{X_0} + \frac{h}{2}\left(\gamma_1\underline{\psi_0}\right) + \frac{h}{2}\gamma_2\left(\underline{f_1} + \frac{\beta h}{3}\left(k(t_1,t_0)\underline{X_0}\right)\right\}\right\}$$

The first and second states are expressed by

$$\overline{X_1} = \left(1 + \frac{h}{2}\gamma_2\,\overline{P_1} - \frac{h^2}{6}\gamma_2\beta k(t_1,t_1)\right\}^{-1}\left(\overline{X_0} + \frac{h}{2}\left(\gamma_1\overline{\psi_0}\right) + \frac{h}{2}\gamma_2\left(\overline{f_1} + \frac{\beta h}{3}\left(k(t_1,t_0)\overline{X_0}\right)\right\}\right\} \tag{12}$$

$$\overline{X_1} = \left\{1 + \frac{h}{2}\gamma_2\overline{P_1} - \frac{h^2}{6}\gamma_2\beta k(t_1,t_1)\right\}^{-1}\left\{\overline{X_0} + \frac{h}{2}\left(\gamma_1\overline{\psi_0}\right) + \frac{h}{2}\gamma_2\left\{\overline{f_1} + \frac{\beta h}{3}\left(k(t_1,t_0)\overline{X_0}\right)\right\}\right\} \tag{13}$$

$$\underline{X_2} = \left(1 + \frac{h}{2}\gamma_2\,\underline{P_2} - \frac{h^2}{6}\gamma_2\beta k(t_2,t_2)\right\}^{-1}\left(\underline{X_1} + \frac{h}{2}\left(\gamma_1\underline{\psi_1}\right) + \frac{h}{2}\gamma_2\left(\underline{f_2} + \frac{\beta h}{3}\left(k(t_2,t_0)\underline{X_0} + 4k(t_2,t_1)\underline{X_1}\right)\right\}\right\}$$

$$\overline{X_2} = \left(1 + \frac{h}{2}\gamma_2\,\overline{P_2} - \frac{h^2}{6}\gamma_2\beta k(t_2,t_2)\right\}^{-1}\left(\overline{X_1} + \frac{h}{2}\left(\gamma_1\underline{\psi_1}\right) + \frac{h}{2}\gamma_2\left(\overline{f_2} + \frac{\beta h}{3}\left(k(t_2,t_0)\overline{X_0} + 4k(t_2,t_1)\overline{X_1}\right)\right\}\right\} \tag{14}$$

4. ILLUSTRATIVE EXAMPLES

To show the efficiency and accuracy considering the proposed method at different step sizes, we look at the following examples:

Example 4.1. *Consider the following integro-differential equations taken from (4)*

$$\begin{cases} X'(t,r) + X(t,r) = ((3+3r)\sinh(t), (8-2r)\sinh(t)) + \int_0^1 (t-s)X(s,r)ds \\ X(0,r) = ((3+3r),(8-2r)), t \in [0,1], 0 \le r \le 1 \end{cases} \tag{15}$$

The exact solution is given by

$$X(t,r) = ((3+3r)\,cosh(t), (8-2r)\,cosh(t)) \tag{16}$$

To compare we use the formula $d(X_n, X(t_n)) = \underset{0\le r\le 1}{Supmax}(\underline{X_n} - \underline{X(t_n)},\ \overline{X_n} - \overline{X(t_n)})$

The approximate solution by using extended difference Euler method is given by

$$\underline{X}_n = \left(1 + \frac{h}{2}\gamma_2\right\}^{-1}\left(\underline{X}_{n-1} + \frac{h}{2}\left(\gamma_1\underline{\psi}_{n-1}\right) + \frac{h}{2}\gamma_2\left((3+3r)\,sinh(t_n) + \frac{\beta h}{3}\left((t_n - t_0)(3+3r) + 4\sum_{k=1}^{n-1}(t_n - t_k)\,\underline{X}_k\right)\right\}\right\}$$

$$\overline{X_n} = \left(1 + \frac{h}{2}\gamma_2\right\}^{-1}\left(\overline{X_{n-1}} + \frac{h}{2}\left(\gamma_1\overline{\psi_{n-1}}\right) + \frac{h}{2}\gamma_2\left((8-2r)\,sinh(t) + \frac{\beta h}{3}\left((t_n - t_0)(8-2r) + 4\sum_{k=1}^{n-1}k\,(t_n - t_k)\,\overline{X_k}\right)\right\}\right\}$$

Approximate solutions $\underline{X}_n$, $\overline{X_n}$ can be found by solving equations in (12) (see Fig. 1., 2, 3, 4, 5, 6) And Table 1, 2, 3)

Table 1. 1 h = 0.01, γ 1 = 1, γ 2 = 1

t	d
0	0
0.3	1.217×10^{-4}
0.5	0.0024
0.7	0.0168
0.9	0.0769

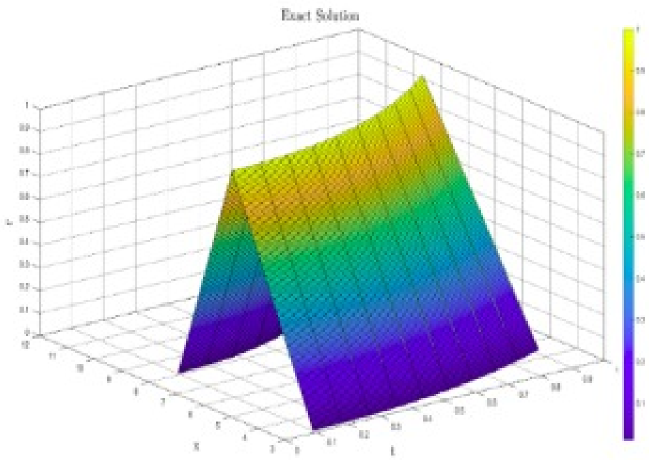

FIGURE 1. Exact Solution

Table 2. h = 0. 01 y 1 = 0. 88, y 2 = 1

t	**d**
0	0
0.3	7.931×10^{-5}
0.5	2.63×10^{-5}
0.7	6.39×10^{-4}
0.9	0.0095

Table 3. h = 0. 01, y 1 = 1, y 2 = 0. 9

t	d
0	0
0.3	3.198×10^{-5}
0.5	1.471×10^{-5}
0.7	0.0018
0.9	0.0160

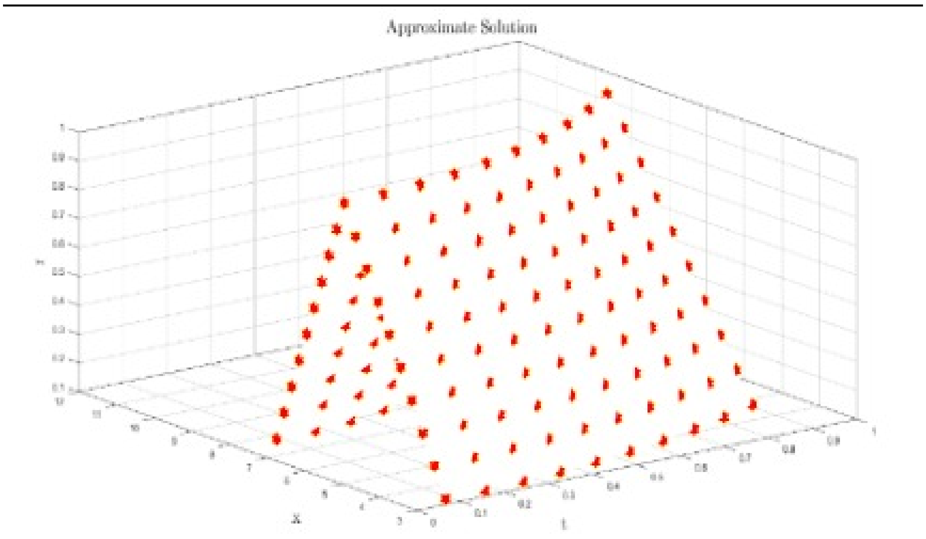

FIGURE 2. Approximate Solution, y 1 = 1, y 2 = 1

Example 4.2. *Consider the following integro-differential equations taken from (4)*

$$\begin{cases} X'(t,r) + 2X(t,r) = ((1+r)(1+t),(3-r)(1+t)) - \int_0^2 X(s,r)ds \\ X(0,r) = (1+r,3-r), t \in [0,2], 0 \le r \le 1 \end{cases} \quad (15)$$

$$P(t,r) = 2$$

$$f(t,r) = ((1+r)(1+t),(3-r)(1+t))$$

$$\beta = -1$$

$$k(t,s) = 1$$

The exact solution is given by

$$X(t,r) = ((1-e^{-t})(1+r) + e^{-t}(1-t)(1+r),(1-e^{-t})(3-r) + e^{-t}(1-t)(3-r)) \quad (16)$$

Approximate solutions X_n , $\overline{X_n}$ by using extended difference Euler method can be found by solving equations in (12) (see Fig. 7., 8, 9) And Table $\bar{4}$, 5, 6)

Table 4. $h = 0.01, \gamma 1 = 1, \gamma 2 = 1$

t	d
0	0
0.3	0.0011
0.**6**	0.0097
0.**9**	0.0288
1. 2	0.0552
1.5	0.0849
1.8	0.1152

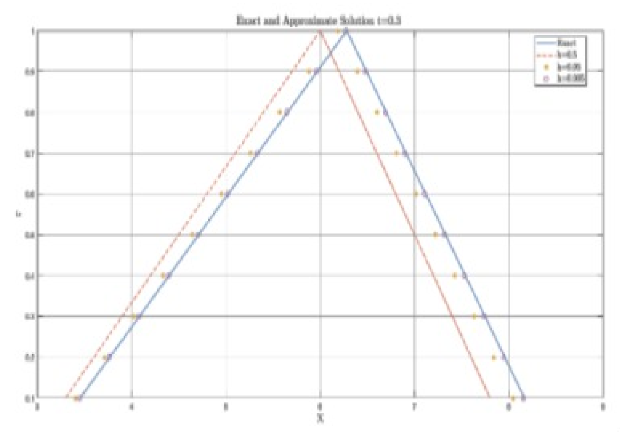

FIGURE 3. **Exact and Approximate Solution at t=0.3**

Table 5. $h = 0.01, \gamma 1 = 0.88, \gamma 2 = 1$

t	d
0	0
0.3	1.188×10^{-5}
0.6	0.0047
0.9	0.0227
1.2	0.0526
1.5	0.0882
1.8	0.1237

Table 6. $h = 0.01, \gamma 1 = 1, \gamma 2 = 0.9$

t	d
0	0
0.3	5.682×10^{-5}
0**. 6**	0.0053
0.**9**	0.0234
1. 2	0.0527
1.5	0.0873

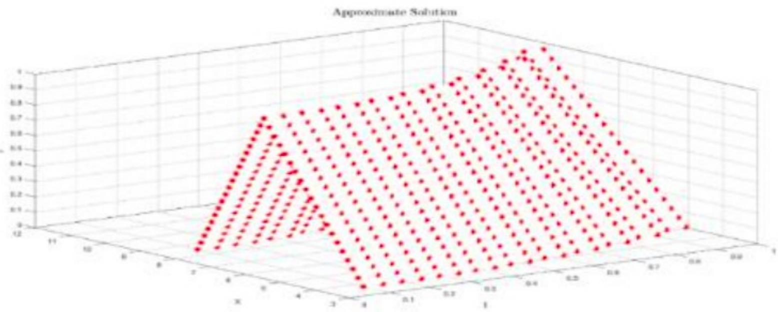

FIGURE 4. **Approximate Solution, y 1 = 0. 88, y 2 = 1**

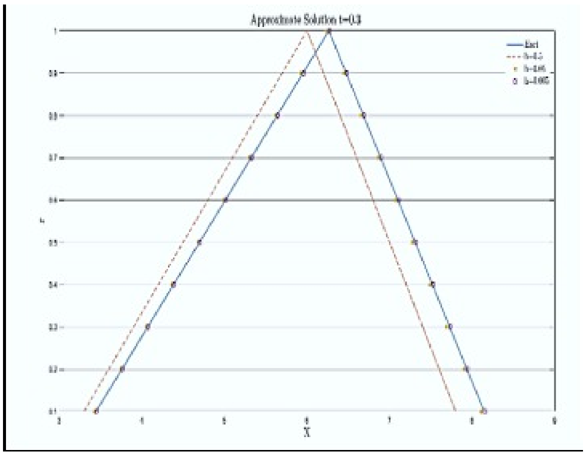

FIGURE 5. **Exact and Approximate Solution at t=0.3**

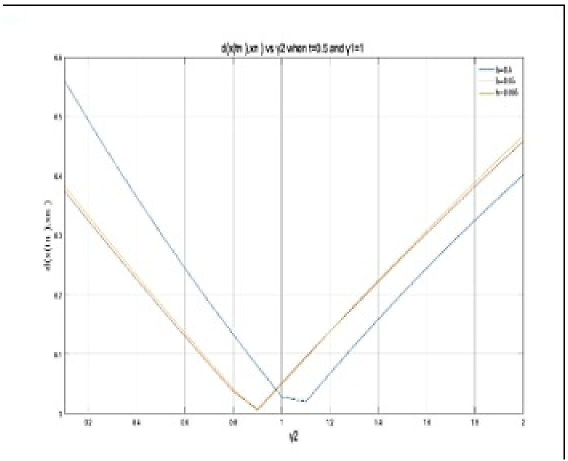

FIGURE 6. **6 $d(X_n, X(t_n))$ vs y 2 when t=0.5**

5. CONCLUSION

Extended difference Euler technique for solving first order fully fuzzy integro-differential Equations was considered. This technique proved its efficient and reliability in solving of these equations by providing the best approximate solutions. The numerical outputs obtained using the proposed technique are comparable to the exact solutions of our proposed model. We showed that the control parameters γ_1 *and* γ_2 played fundamental and important role in reducing the error rate which resulting from the approximation of solutions for fuzzy integro-differential Equations so, Extended Euler method is more accurate in terms of absolute error.

Thus, our work in this paper, can be extended to multivariate fuzzy equations. Finally, we would like to refer that the proposed equation can be applied to real models and used for data analysis in various systems such as medicine, economy, engineering, biomedical, and environmental.

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