Characterizations of $\mathfrak{sf}$ -simple extension of topologies

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ABSTRACT: The main aim of this study is to create a new type of topology called $\mathfrak{sf}$ -simple extension and investigate its properties. We introduce a new definition for $\mathfrak{sf}$ -open sets and consider this aspect as the basis of our main definition. Furthermore, we investigate the properties of the proposed concept to allow us to provide new examples of explicit descriptions of topological spaces and certain types of $\mathfrak{sf}$ -covering for topological spaces, such as ($\mathfrak{sf}$ -Lindelof and $\mathfrak{sf}$ -paracompact spaces). The use of the tool offers important results for topological spaces. Other findings related to the proposed approach have also been identified.

Keywords: $\mathfrak{sf}$ -open set, $\mathfrak{sf}$ -paracompact space, $\mathfrak{sf}$ -Lindelof space, $\mathfrak{sf}$ -regular space, $\mathfrak{sf}$ -normal, $\mathfrak{sf}$ -simple.

1. INTRODUCTION

In the domain of general topological spaces ($\text{top-}\mathfrak{s}$'s), the notion of semi-feebly was first introduced by Levine [1]. A set $\mathcal{Q}$ of a $\text{top-}\mathfrak{s}$ ($\mathfrak{X}, \varphi$) is a semi-open ($\mathfrak{s}$ -open) set if there exists an open set $\mathcal{O}$ of $\mathfrak{X}$ such that $\mathcal{O} \subseteq \mathcal{Q} \subseteq \text{cl}(\mathcal{O})$, where $\text{cl}(\mathcal{O})$ is the closure of $\mathcal{O}$. This concept has played an important role in the research of $\text{top-}\mathfrak{s}$'s. The appropriateness of the definition is depicted by the capability of researchers (e.g., Refs. [2] and [1]) to extend all basic results of $\mathfrak{s}$ -open sets to the generalized $\text{top-}\mathfrak{s}$'s.

In Refs. [2] and [3], $\mathfrak{s}$ -closure of sets were used to define and investigate feebly open sets ($\mathfrak{f}$ -open sets), feebly closed sets ($\mathfrak{f}$ -closed sets), and $\mathfrak{f}$ -closure of sets. Maheshwari and Tapi [4] defined a set $\mathcal{Q}$ of a $\text{top-}\mathfrak{s}$ ($\mathfrak{X}, \varphi$) as a feebly-open ($\mathfrak{f}$ -open) set if there exists an open set $\mathcal{O}$ of $\mathfrak{X}$ such that $\mathcal{O} \subseteq \mathcal{Q} \subseteq \text{cl}_{\mathfrak{s}}(\mathcal{O})$, where $\text{cl}_{\mathfrak{s}}(\mathcal{O})$ is the $\mathfrak{s}$ -closure of $\mathcal{O}$. The complement of the $\mathfrak{f}$ -open set is the $\mathfrak{f}$ -closed set. The $\mathfrak{f}$ -closure of $\mathcal{Q}$, denoted by $\text{cl}_{\mathfrak{f}}(\mathcal{Q})$, is the intersection of all $\mathfrak{f}$ -closed sets containing $\mathcal{Q}$. The collection of all $\mathfrak{f}$ -open sets in $\text{top-}\mathfrak{s}$ ($\mathfrak{X}, \varphi$) is a top on $\mathfrak{X}$ [5].

In Refs. [6] and [7], several properties of $\text{top-}\mathfrak{s}$'s were investigated and characterized using the concept of $\mathfrak{f}$ -open sets. In 2020, Othman and Raad [8–10] combined two concepts ($\mathfrak{s}$ -open and $\mathfrak{f}$ -open concepts) to produce a new concept of open sets, namely, the semi-feebly open ($\mathfrak{sf}$ -open) sets, and subsequently investigated their basic properties. They also established some of the relationships between this new class of sets and the spaces (e.g., $\mathfrak{sf}$ -normal spaces). In 2021, Zahra and Raad [11, 12] introduced new types of topological spaces named the $\mathfrak{sf}'$ -regular and $\mathfrak{sf}''$ -normal spaces, subsequently offering a complete characterization of these spaces after studying their properties. Recall that a $\text{top-}\mathfrak{s}$ ($\mathfrak{X}, \varphi$) satisfies a conduct union if the union of all $\mathfrak{sf}$ -open sets in $\mathfrak{X}$ is also $\mathfrak{sf}$ -open [8].

This research is devoted to the study of the relation between certain properties of $\mathfrak{sf}$ -open sets in a $\text{top-}\mathfrak{s}$ by adopting a new structure that is a generalization of a family of $\mathfrak{sf}$ -open sets. More specifically, we study a new type of space, namely, the $\mathfrak{sf}$ -simple extension space, in which a conduct union is a necessary condition.

2. BACKGROUND MATERIAL:

In this section, we recollect a few basic conceptions and introduce notations for understanding $\mathfrak{sf}$ -open sets.

Definition 2.1. [8]: Let $\mathfrak{X}$ be a top- $\mathfrak{s}$ and $\mathfrak{Q}$ be a subset of $\mathfrak{X}$. Then, is

- i. a $\mathfrak{s}\mathfrak{f}$ -open set if $\mathfrak{Q} \subseteq \mathfrak{O}$, where $\mathfrak{O}$ is a $\mathfrak{s}$ -open set. Then, $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \subseteq \mathfrak{O}$.
- ii. a $\mathfrak{s}\mathfrak{f}$ -closed set if $\mathfrak{C} \subseteq int_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$, where $\mathfrak{C}$ is a $\mathfrak{s}$ -closed set.

Remark 2.2. [8]:

- i. Each closed ($\mathfrak{f}$ -closed) set is $\mathfrak{s}\mathfrak{f}$ -open.
- ii. The arbitrary intersection of $\mathfrak{s}\mathfrak{f}$ -open sets is $\mathfrak{s}\mathfrak{f}$ -open.

Definition 2.3. [8]: Let $(\mathfrak{X}, \varphi)$ be a top- $\mathfrak{s}$ and $\mathfrak{Q} \subseteq \mathfrak{X}$. Then,

- i. a set $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$ is a $\mathfrak{s}\mathfrak{f}$ -closure of $\mathfrak{Q}$, which is defined by

$$cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) = \bigcap \{ \mathfrak{C} : \mathfrak{C} \text{ is a } \mathfrak{s}\mathfrak{f}\text{-closed set in } \mathfrak{X}; \mathfrak{Q} \subseteq \mathfrak{C} \}. \quad (1)$$

- ii. a set $int_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$ is a $\mathfrak{s}\mathfrak{f}$ -interior of $\mathfrak{Q}$, which is defined by

$$int_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) = \bigcup \{ \mathfrak{O} : \mathfrak{O} \text{ is a } \mathfrak{s}\mathfrak{f}\text{-open set in } \mathfrak{X}; \mathfrak{O} \subseteq \mathfrak{Q} \}. \quad (2)$$

Proposition 2.4. [8]: Let $(\mathfrak{X}, \varphi)$ be a top- $\mathfrak{s}$ satisfying a conduct union. Thus,

- i. $cl_{\mathfrak{s}\mathfrak{f}}(\emptyset) = \emptyset$ and $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{X}) = \mathfrak{X}$.
- ii. $\mathfrak{Q} \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$ for all $\mathfrak{Q} \subseteq \mathfrak{X}$.
- iii. if $\mathfrak{Q} \subseteq \mathfrak{S}$, then $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$, where $\mathfrak{Q}, \mathfrak{S} \subseteq \mathfrak{X}$.
- iv. $cl_{\mathfrak{s}\mathfrak{f}}(cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})) = cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$ for all $\mathfrak{Q} \subseteq \mathfrak{X}$.

Theorem 2.5. [8]: Let $(\mathfrak{X}, \varphi)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathfrak{Q} \subseteq \mathfrak{X}$, where $\mathfrak{x} \in \mathfrak{X}$. Then,

- i. $\mathfrak{x} \in cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$ if and only if $\mathfrak{Q} \cap \mathfrak{O} \neq \emptyset$ for each $\mathfrak{s}\mathfrak{f}$ -open set $\mathfrak{O}$ in $\mathfrak{X}$ containing $\mathfrak{x}$.
- ii. a set $\mathfrak{Q}$ is $\mathfrak{s}\mathfrak{f}$ -closed if and only if $\mathfrak{Q} = cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$.

Proposition 2.6. Let $(\mathfrak{X}, \varphi)$ be a top- $\mathfrak{s}$ satisfying conduct union. Then,

- i. $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cap \mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cap cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$ for all $\mathfrak{Q}, \mathfrak{S} \subseteq \mathfrak{X}$.
- ii. $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cup \mathfrak{S}) = cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cup cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$ for all $\mathfrak{Q}, \mathfrak{S} \subseteq \mathfrak{X}$.

Proof. i. $\mathfrak{Q} \cap \mathfrak{S} \subseteq \mathfrak{Q}$ and $\mathfrak{Q} \cap \mathfrak{S} \subseteq \mathfrak{S}$. According to Proposition (2.4.iii),

$$cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cap \mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \text{ and } cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cap \mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S}). \quad (3)$$

Therefore, $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cap \mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cap cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$.

ii. $\mathfrak{Q} \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q})$ and $\mathfrak{S} \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$; thus, $\mathfrak{Q} \cup \mathfrak{S} \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cup cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$. According to Proposition (2.4.iii) again, it follows that $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cup \mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cup \mathfrak{S})$. However, $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cup \mathfrak{S})$ is the smallest $\mathfrak{s}\mathfrak{f}$ -closed set of $\mathfrak{X}$ containing $\mathfrak{Q} \cup \mathfrak{S}$.

Therefore, $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cup \mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cup cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$. By applying the same technique in which $\mathfrak{Q} \subseteq \mathfrak{Q} \cup \mathfrak{S}$ and $\mathfrak{S} \subseteq \mathfrak{Q} \cup \mathfrak{S}$, we have

$$cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cup cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cup \mathfrak{S}) \quad (4)$$

Remark 2.7. The converse of (i) in Proposition (2.6) cannot be regarded as true in general. For example, let $\mathfrak{X} = \mathbb{Z}$ be a set of all integer numbers with φ_{ind} as a top on $\mathfrak{X}$. Thus, $cl_{\mathfrak{s}\mathfrak{f}}(\mathbb{Z}_o \cap \mathbb{Z}_e) = \emptyset \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathbb{Z}_o) \cap cl_{\mathfrak{s}\mathfrak{f}}(\mathbb{Z}_e) = \mathfrak{X}$, but $cl_{\mathfrak{s}\mathfrak{f}}(\mathbb{Z}_o) \cap cl_{\mathfrak{s}\mathfrak{f}}(\mathbb{Z}_e) \not\subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathbb{Z}_o \cap \mathbb{Z}_e)$.

Proposition 2.8. Let $(\mathfrak{X}, \varphi)$ be a top- $\mathfrak{s}$ satisfying a conduct union. Then, a finite intersection of $\mathfrak{s}\mathfrak{f}$ -closed sets is $\mathfrak{s}\mathfrak{f}$ -closed.

Proof. By induction, $n = 2$. Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two $\mathfrak{s}\mathfrak{f}$ -closed sets of $\mathfrak{X}$ for demonstrating $\mathfrak{C}_1 \cap \mathfrak{C}_2$ as a $\mathfrak{s}\mathfrak{f}$ -closed set. According to Theorem (2.5.ii),

$$\mathfrak{C}_1 = cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{C}_1) \text{ and } \mathfrak{C}_2 = cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{C}_2) \text{ because } \mathfrak{C}_1 \cap \mathfrak{C}_2 = cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cap cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cap \mathfrak{S}). \quad (5)$$

However, $cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q} \cap \mathfrak{S}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{Q}) \cap cl_{\mathfrak{s}\mathfrak{f}}(\mathfrak{S})$. In the same manner, according to Theorem (2.5.ii), we have $\mathfrak{C}_1 \cap \mathfrak{C}_2$, which is a $\mathfrak{s}\mathfrak{f}$ -closed set.

Proposition 2.9. [8, 8]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union. Thus,

- i. $\text{int}_{\mathfrak{sf}}(\emptyset) = \emptyset$ and $\text{int}_{\mathfrak{sf}}(\mathfrak{X}) = \mathfrak{X}$.
- ii. $\text{int}_{\mathfrak{sf}}(\mathcal{Q}) \subseteq \mathcal{Q}$ for all $\mathcal{Q} \subseteq \mathfrak{X}$.
- iii. if $\mathcal{Q} \subseteq \mathcal{S}$, then $\text{int}_{\mathfrak{sf}}(\mathcal{Q}) \subseteq \text{int}_{\mathfrak{sf}}(\mathcal{S})$ for all $\mathcal{Q}, \mathcal{S} \subseteq \mathfrak{X}$.
- iv. $\text{int}_{\mathfrak{sf}}(\text{int}_{\mathfrak{sf}}(\mathcal{Q})) = \text{int}_{\mathfrak{sf}}(\mathcal{Q})$ for all $\mathcal{Q} \subseteq \mathfrak{X}$.

Theorem 2.10. [10]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$, where $\mathfrak{x} \in \mathfrak{X}$. Then,

- i. $\mathfrak{x} \in \text{int}_{\mathfrak{sf}}(\mathcal{Q})$ if and only if there exists a $\mathfrak{sf}$ -open set $\mathcal{O}$ in $\mathfrak{X}$ containing $\mathfrak{x}$ such that $\mathfrak{x} \in \mathcal{O} \subseteq \mathcal{Q}$.
- ii. a set $\mathcal{Q}$ is $\mathfrak{sf}$ -open if and only if $\mathcal{Q} = \text{int}_{\mathfrak{sf}}(\mathcal{Q})$.

Definition 2.11. [10]: Let $(\mathfrak{Y}, \wp_{\mathfrak{Y}})$ be a subspace of a top- $\mathfrak{s}$ $(\mathfrak{X}, \wp)$. Thus, $\mathcal{Q} \subseteq \mathfrak{Y}$ is a $\mathfrak{sf}$ -open set of $\mathfrak{Y}$. If $\mathcal{Q} \subseteq \mathcal{O}$, where $\mathcal{O}$ is a $\mathfrak{s}$ -open set of $\mathfrak{Y}$, then $[\text{cl}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{Y}} \subseteq \mathcal{O}$, where $[\text{cl}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{Y}}$ is a $\mathfrak{sf}$ -closure of $\mathcal{Q}$ in $(\mathfrak{Y}, \wp_{\mathfrak{Y}})$.

Theorem 2.12. [8]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$, and let $\emptyset \neq \mathfrak{Y} \subseteq \mathfrak{X}$. Then, for any $\mathcal{Q} \subseteq \mathfrak{Y}$, it follows that

- i. $[\text{int}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{X}} \subseteq [\text{int}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{Y}} \cap \mathfrak{Y}$.
- ii. $[\text{cl}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{Y}} = [\text{cl}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{X}} \cap \mathfrak{Y}$.

Theorem 2.13. : Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\emptyset \neq \mathfrak{Y} \subseteq \mathfrak{X}$.

- i. For any $\mathcal{Q} \subseteq \mathfrak{Y}$, then $\mathcal{Q} \cap \mathcal{O}$ is a $\mathfrak{sf}$ -open set of $\mathfrak{Y}$, where $\mathcal{O}$ is a $\mathfrak{sf}$ -open set of $\mathfrak{X}$.
- ii. For any $\mathcal{Q} \subseteq \mathfrak{Y}$, then $\mathcal{Q} \cap \mathcal{C}$ is a $\mathfrak{sf}$ -closed set of $\mathfrak{Y}$, where $\mathcal{C}$ is a $\mathfrak{sf}$ -closed set of $\mathfrak{X}$.

Proof. i. $\mathcal{O}$ is a $\mathfrak{sf}$ -open set of $\mathfrak{X}$. According to Theorem (2.10.ii), $\mathcal{O} = [\text{int}_{\mathfrak{sf}}(\mathcal{O})]_{\mathfrak{X}}$. Furthermore, according to Theorem (2.12.i), it also follows that

$$[\text{int}_{\mathfrak{sf}}(\mathcal{O})]_{\mathfrak{X}} \subseteq [\text{int}_{\mathfrak{sf}}(\mathcal{O})]_{\mathfrak{Y}} \cap \mathfrak{Y}. \quad (6)$$

Hence,

$$\mathcal{O} \subseteq [\text{int}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{Y}} \cap \mathfrak{Y}. \quad (7)$$

$$[\text{int}_{\mathfrak{sf}}(\mathcal{Q})]_{\mathfrak{Y}} = \bigcup \{ \mathcal{U} : \mathcal{U} \text{ is } \mathfrak{sf}\text{-open in } \mathfrak{Y} \text{ and } \mathcal{U} \subseteq \mathcal{Q} \} \quad (8)$$

Consequently, according to Theorem (2.10.ii), $\mathcal{Q} \cap \mathcal{O}$ is a $\mathfrak{sf}$ -open set of $\mathfrak{Y}$. The proof of (ii) can be deduced directly in the same manner.

Theorem 2.14. [8]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$, and let $\mathcal{Q} \subseteq \mathfrak{X}$. Then,

- i. $\text{cl}_{\mathfrak{sf}}(\mathcal{Q}) = (\text{int}_{\mathfrak{sf}}(\mathcal{Q}^c))^c$.
- ii. $\text{int}_{\mathfrak{sf}}(\mathcal{Q}) = (\text{cl}_{\mathfrak{sf}}(\mathcal{Q}^c))^c$.

Definition 2.15. [11]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$. Then, $\mathfrak{X}$ is a $\mathfrak{sf}''$ -regular space for each $\mathfrak{x} \in \mathfrak{X}$ in $\mathfrak{X}$, and $\mathcal{F}$ is a $\mathfrak{sf}$ -closed subset such that if $\mathfrak{x} \notin \mathcal{F}$, there exist disjoint $\mathfrak{sf}$ -open sets $\mathcal{U}$ and $\mathcal{V}$, with $\mathfrak{x} \in \mathcal{U}$ and $\mathcal{F} \subseteq \mathcal{V}$.

Theorem 2.16. [11]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$. Then, $\mathfrak{X}$ is a $\mathfrak{sf}''$ -regular space if and only if for each $\mathfrak{x} \in \mathfrak{X}$ and for each $\mathfrak{sf}$ -open set $\mathcal{U}$ of $\mathfrak{X}$ with $\mathfrak{x} \in \mathcal{U}$, there exists an $\mathfrak{sf}$ -open set $\mathcal{V}$ of $\mathfrak{X}$ such that $\mathfrak{x} \in \mathcal{V} \subseteq \text{cl}_{\mathfrak{sf}}(\mathcal{V}) \subseteq \mathcal{U}$.

Definition 2.17. [12]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ and $\mathcal{Q} \subseteq \mathfrak{X}$. Then, $\mathcal{Q}$ is a $\mathfrak{sf}$ -dense subset if $\text{cl}_{\mathfrak{sf}}(\mathcal{Q}) = \mathfrak{X}$, as denoted by $(\mathfrak{sf}\text{-}\mathfrak{d}\mathfrak{n}\mathfrak{s})$.

Theorem 2.18. [8]: Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union. Then, $(\mathfrak{X}, \wp)$ is a $\mathfrak{sf}''$ -normal space if and only if for each $\mathfrak{sf}$ -closed set $\mathcal{D}$ of $\mathfrak{X}$ and for each $\mathfrak{sf}$ -open set $\mathcal{O}$ of $\mathfrak{X}$ with $\mathcal{D} \subseteq \mathcal{O}$, there exists a $\mathfrak{sf}$ -open set $\mathcal{V}$ with $\mathcal{D} \subseteq \mathcal{V} \subseteq \text{cl}_{\mathfrak{sf}}(\mathcal{V}) \subseteq \mathcal{O}$.

Definition 2.19. [4]: The family $\{\mathcal{A}_{\lambda} : \lambda \in \Lambda\}$ of subsets of top- $\mathfrak{s}$ $(\mathfrak{X}, \wp)$ is $\mathfrak{sf}$ -locally finite if for each point $\mathfrak{x} \in \mathfrak{X}$, there exists $\mathcal{U} \in \mathfrak{N}_{\mathfrak{sf}}(\mathfrak{x})$ such that $\{\lambda \in \Lambda : \mathcal{U} \cap \mathcal{A}_{\lambda} \neq \emptyset\}$ is finite.

Definition 2.20. [4]: A top- $\mathfrak{s}$ $(\mathfrak{X}, \wp)$ is a $\mathfrak{sf}$ -paracompact if each $\mathfrak{sf}$ -open cover of $\mathfrak{X}$ entails a $\mathfrak{sf}$ -locally finite $\mathfrak{sf}$ -open refinement.

Definition 2.21. [8]: A top- $\mathfrak{s}$ $(\mathfrak{X}, \wp)$ is $\mathfrak{sf}$ -Lindelof if each $\mathfrak{sf}$ -open cover of $\mathfrak{X}$ has a countable subcover.

3. MAIN RESULTS

In this section, we investigate a number of results concerning the $\mathfrak{s}\mathfrak{f}$ -simple extension of topological spaces. We concentrate mainly on the properties of some types of topological spaces, such as $\mathfrak{s}\mathfrak{f}''$ -regular, $\mathfrak{s}\mathfrak{f}''$ -normal, and $\mathfrak{s}\mathfrak{f}$ -Lindelof spaces.

Definition 3.1. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$. Then,

$$\wp(\mathcal{Q}) = \{\mathcal{U} \cup (\mathcal{V} \cap \mathcal{Q}) : \mathcal{U} \text{ and } \mathcal{V} \text{ are } \mathfrak{s}\mathfrak{f}\text{-open sets of } \mathfrak{X}\}, \quad (9)$$

which is a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$.

Example 3.2. Let $\mathfrak{X} = \{a, b, c\}$ include top $\wp = \{\emptyset, \mathfrak{X}, \{a\}\}$. Then, $\wp^{\mathfrak{s}\mathfrak{f}} = \{\emptyset, \mathfrak{X}, \{b\}, \{c\}, \{b, c\}\}$ and $(\mathfrak{X}, \wp)$ satisfy a conduct union. Assume the set $\mathcal{Q} = \{a, c\}$. Then, $\wp(\mathcal{Q}) = \{\emptyset, \mathfrak{X}, \mathcal{Q}, \{b\}, \{c\}, \{b, c\}\}$.

Remark 3.3. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. Then,

- i. $\wp \subseteq \wp(\mathcal{Q})$.
- ii. $\wp(\mathcal{Q})$ forms a top on $\mathfrak{X}$.

Remark 3.4. The conduct union for $(\mathfrak{X}, \wp)$ is a necessary condition in Definition (3.1), enabling $\wp(\mathcal{Q})$ to be a top on $\mathfrak{X}$, as shown by Example 3.5.

Example 3.5. Let $\mathfrak{X} = \{a, b, c\}$ with $\wp = \{\emptyset, \mathfrak{X}, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be a top on $\mathfrak{X}$. Then, $\{\emptyset, \mathfrak{X}, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$ is the set of all $\mathfrak{s}\mathfrak{f}$ -open sets of $\mathfrak{X}$. Clearly, $(\mathfrak{X}, \wp)$ does not satisfy a conduct union.

Assume the set $\mathcal{Q} = \{c\}$. Then, $\wp(\mathcal{Q}) = \{\emptyset, \mathfrak{X}, \mathcal{Q}, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$ is not $\wp(\mathcal{Q})$ as a top on $\mathfrak{X}$.

Notation 3.6. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. Then,

- i. $\wp(\mathcal{Q})$ $\mathfrak{s}\mathfrak{f}$ -open denotes a $\mathfrak{s}\mathfrak{f}$ -open with respect to $\wp(\mathcal{Q})$.
- ii. $\wp(\mathcal{Q})$ $\mathfrak{s}\mathfrak{f}$ -closed denotes a $\mathfrak{s}\mathfrak{f}$ -closed with respect to $\wp(\mathcal{Q})$.
- iii. $\wp(\mathcal{Q}) \text{int}_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})$ denotes a $\mathfrak{s}\mathfrak{f}$ -interior of $\mathcal{Q}$ with respect to $\wp(\mathcal{Q})$.
- iv. $\wp(\mathcal{Q}) \text{cl}_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})$ denotes a $\mathfrak{s}\mathfrak{f}$ -closure of $\mathcal{Q}$ with respect to $\wp(\mathcal{Q})$.
- v. $\wp(\mathcal{Q}) \text{nbhd}_{\mathfrak{s}\mathfrak{f}}(x)$ denotes a $\mathfrak{s}\mathfrak{f}$ -nbhd of x with respect to $\wp(\mathcal{Q})$.

Proposition 3.7. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. Then,

- i. $\mathcal{Q}$ is $\wp(\mathcal{Q})$ $\mathfrak{s}\mathfrak{f}$ -open.
- ii. $\mathcal{Q}^c$ is $\wp(\mathcal{Q})$ $\mathfrak{s}\mathfrak{f}$ -closed.

Proof. i. According to Definition (3.1),

$$\wp(\mathcal{Q}) = \{\mathcal{U} \cup (\mathcal{V} \cap \mathcal{Q}) : \mathcal{U} \text{ and } \mathcal{V} \text{ are } \mathfrak{s}\mathfrak{f}\text{-open sets of } \mathfrak{X}\}. \quad (10)$$

If we take $\mathcal{U} = \emptyset$ and $\mathcal{V} = \mathfrak{X}$, it follows that $\mathcal{Q} \in \wp(\mathcal{Q})$. Then, $\mathcal{Q}$ is a $\wp(\mathcal{Q})$ $\mathfrak{s}\mathfrak{f}$ -open set.

ii. Clearly.

Proposition 3.8. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. Then,

- i. $\wp \cap \mathcal{Q} = \wp(\mathcal{Q}) \cap \mathcal{Q}$.
- ii. $\wp \cap \mathcal{Q}^c = \wp(\mathcal{Q}) \cap \mathcal{Q}^c$.

Proof. i. According to Remark (3.3.i),

$$\wp \cap \mathcal{Q} \subseteq \wp(\mathcal{Q}) \cap \mathcal{Q} \quad (11)$$

To further prove the containment, let $\mathcal{W} \in \wp(\mathcal{Q}) \cap \mathcal{Q}$. Then, $\mathcal{W} \in \wp(\mathcal{Q})$ and $\mathcal{W} \subseteq \mathcal{Q}$. Consequently, $\mathcal{W} = \mathcal{U} \cup (\mathcal{V} \cap \mathcal{Q})$ for some $\mathfrak{s}\mathfrak{f}$ -open sets $\mathcal{U}$ and $\mathcal{V}$ of $\mathfrak{X}$. However, $\mathcal{W} \subseteq \mathcal{Q}$ with

$$\wp \cap \mathcal{Q} = \{\mathcal{O} \cap \mathcal{Q} : \mathcal{O} \text{ is } \mathfrak{s}\mathfrak{f}\text{-open set of } \mathfrak{X}\}, \quad (12)$$

and we have $\mathcal{U} \subseteq \mathcal{Q}$. Thus, $\mathcal{W} \in \wp(\mathcal{Q}) \cap \mathcal{Q}$, and $\wp(\mathcal{Q}) \cap \mathcal{Q} \subseteq \wp \cap \mathcal{Q}$ such that

$$\wp \cap \mathcal{Q} = \wp(\mathcal{Q}) \cap \mathcal{Q}. \quad (13)$$

ii. Applying the same technique as that in (i) for the proof allows for the obtainment of the desired result.

Proposition 3.9. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q}, C \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. Then,

- i. $\wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(C) = cl_{\mathfrak{s}\mathfrak{f}}(C) \cap (\mathcal{Q}^c \cup (\mathcal{Q} \cap [cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}}))$.
- ii. $\wp(\mathcal{Q})int_{\mathfrak{s}\mathfrak{f}}(C) = int_{\mathfrak{s}\mathfrak{f}}(C) \cup [int_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}}$.
- iii. $\wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q}) = cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})$.

Proof. i. According to Theorem (2.14.i),

$$\begin{aligned} \wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(C) &= (int_{\mathfrak{s}\mathfrak{f}}(C^c))^c \\ &= ((int_{\mathfrak{s}\mathfrak{f}}(C^c) \cup [int_{\mathfrak{s}\mathfrak{f}}(C^c)]_{\mathcal{Q}} \cap \mathcal{Q})^c \\ &= (int_{\mathfrak{s}\mathfrak{f}}(C^c))^c \cap ([int_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}})^c. \\ &= cl_{\mathfrak{s}\mathfrak{f}}(C) \cap (\mathcal{Q}^c \cup [int_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}})^c. \\ &= cl_{\mathfrak{s}\mathfrak{f}}(C) \cap (\mathcal{Q}^c \cup [cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}}). \\ &= cl_{\mathfrak{s}\mathfrak{f}}(C) \cap (\mathcal{Q}^c \cup [cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}}) \end{aligned} \quad (14)$$

ii. According to Theorem (2.14.ii) and (i) above, we can obtain the required condition.

iii.

$$\wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q}) = cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q}) \cap (\mathcal{Q}^c \cup (\mathcal{Q} \cap cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q}))). \quad (15)$$

According to (ii),

$$\wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q}) = cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q}). \quad (16)$$

Remark 3.10. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. A basis of $\wp(\mathcal{Q})$ on $\mathfrak{X}$ is the family $\mathfrak{B}(\mathfrak{X}) = \wp \cup \wp_{\mathcal{Q}}$.

Theorem 3.11. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. Then, $\mathcal{Q}$ is a $\mathfrak{s}\mathfrak{f}$ -closed set if and only if it is $\wp(\mathcal{Q})$ $\mathfrak{s}\mathfrak{f}$ -closed.

Proof. According to Proposition (3.9.i), $C = \mathcal{Q}$, and it follows that $\wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) = cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})$ and $\mathcal{Q} = \wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})$ if and only if $\mathcal{Q} = cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})$.

Theorem 3.12. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$ and $\wp(\mathcal{Q})$ be a $\mathfrak{s}\mathfrak{f}$ -simple extension of $\wp^{\mathfrak{s}\mathfrak{f}}$. If $\mathfrak{x} \notin \mathcal{Q}$ and $C \subseteq \mathfrak{X}$, then $\mathfrak{x} \in cl_{\mathfrak{s}\mathfrak{f}}(C)$ if and only if $\mathfrak{x} \in \wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})$.

Proof. According to Proposition (3.9.i),

$$\wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(C) = cl_{\mathfrak{s}\mathfrak{f}}(C) \cap (\mathcal{Q}^c \cup (\mathcal{Q} \cap [cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}})). \quad (17)$$

If $\mathfrak{x} \in \wp(\mathcal{Q})cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})$, then $\mathfrak{x} \in cl_{\mathfrak{s}\mathfrak{f}}(C)$ because $\mathfrak{x} \in cl_{\mathfrak{s}\mathfrak{f}}(C) \cap (\mathcal{Q}^c \cup (\mathcal{Q} \cap [cl_{\mathfrak{s}\mathfrak{f}}(C \cap \mathcal{Q})]_{\mathcal{Q}}))$. Moreover, from Remark (3.3.i) and Theorem (2.5.i), we can obtain the required condition.

Theorem 3.13. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathcal{S} \subseteq \mathfrak{X}$ such that $\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) = \mathcal{S} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{S})$. If $cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{S} \setminus \mathcal{Q}) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = \emptyset$ and $cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{S} \setminus \mathcal{Q}) \cap (\mathcal{S} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{S})) = \emptyset$, for any $\mathfrak{s}\mathfrak{f}$ -open set $\mathcal{U}$ of $\mathfrak{X}$ with $\mathcal{U} \setminus cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)$ being a $\mathfrak{s}\mathfrak{f}$ -open set $\mathcal{U}$ of $\mathfrak{X}$, then

- i. $\mathcal{U} \cap int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) = (\mathcal{U} \setminus cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap int_{\mathfrak{s}\mathfrak{f}}(\mathcal{S})$.
- ii. $\mathcal{U} \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = (\mathcal{U} \setminus cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}))$.

Proof. i. Let $x \in \mathcal{U}$ and $x \in \text{int}_{\text{sf}}(\mathcal{Q}) \subseteq \text{int}_{\text{sf}}(\mathcal{S})$.

According to Theorem (2.14.i), if $x \in \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c) \subseteq \text{cl}_{\text{sf}}(\mathcal{U}) \cap \text{cl}_{\text{sf}}(\mathcal{S}) \cap \text{cl}_{\text{sf}}(\mathcal{Q}^c)$, then $x \in (\text{int}_{\text{sf}}(\mathcal{Q}))^c$, where $x \notin \text{int}_{\text{sf}}(\mathcal{Q})$. However, this definition is irrational. Thus, $x \in (\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \text{int}_{\text{sf}}(\mathcal{S})$.

Then, $\mathcal{U} \cap \text{int}_{\text{sf}}(\mathcal{Q}) \subseteq (\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \text{int}_{\text{sf}}(\mathcal{S})$.

For the proof of $(\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \text{int}_{\text{sf}}(\mathcal{S}) \subseteq \mathcal{U} \cap \text{int}_{\text{sf}}(\mathcal{Q})$, let us preliminarily consider $\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c) \subseteq \mathcal{U}$. Then, we have

$$\begin{aligned} (\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \text{int}_{\text{sf}}(\mathcal{S}) &= \mathcal{U} \cap \text{int}_{\text{sf}}(\mathcal{U}^c \cup \mathcal{S} \cup \mathcal{Q}) \cap \text{int}_{\text{sf}}(\mathcal{S}) \\ &= \mathcal{U} \cap \text{int}_{\text{sf}}(\mathcal{U}^c \cap \mathcal{S} \cup \mathcal{Q}). \end{aligned} \quad (18)$$

Similarly, $(\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \text{int}_{\text{sf}}(\mathcal{Q}) = \mathcal{U} \cap \text{int}_{\text{sf}}(\mathcal{U}^c \cup \mathcal{Q}) \cap \text{int}_{\text{sf}}(\mathcal{Q})$.

ii. The assertion can be explained by

$$\begin{aligned} \mathcal{U} \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) &= (\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{S} \cap \mathcal{Q}^c)) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) \\ &\subseteq (\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})). \end{aligned} \quad (19)$$

Theorem 3.14. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q}, \mathcal{S} \subseteq \mathfrak{X}$. Then, $\wp(\mathcal{Q}) = \wp(\mathcal{S})$ if and only if

i. $\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}) = \mathcal{S} \setminus \text{int}_{\text{sf}}(\mathcal{S})$.

ii. $\text{cl}_{\text{sf}}(\mathcal{S} \setminus \mathcal{Q}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) = \text{cl}_{\text{sf}}(\mathcal{Q} \setminus \mathcal{S}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) = \emptyset$.

Proof. Assume that (i) and (ii) have been verified. To demonstrate that $\wp(\mathcal{Q}) = \wp(\mathcal{S})$, suppose preliminarily that $\mathcal{S} \subseteq \mathcal{Q}$. Then, $\wp(\mathcal{Q}) \subseteq \wp(\mathcal{S})$. According to Proposition (3.9.ii), $\mathcal{W} = \mathcal{U} \cap \mathcal{Q} = (\mathcal{U} \cap \text{int}_{\text{sf}}(\mathcal{Q})) \cup (\mathcal{U} \cap \mathcal{S}) \in \wp(\mathcal{Q})$ for each sf-open set $\mathcal{U}$ of $\mathfrak{X}$.

Thus, $\wp(\mathcal{Q}) = \wp(\mathcal{S})$, and $\wp(\mathcal{S}) \subseteq \wp(\mathcal{Q})$. According to Theorem (2.13), for each sf-open set $\mathcal{U}$ of $\mathfrak{X}$, we have

$$\begin{aligned} \mathcal{V} = \mathcal{U} \cap \mathcal{S} &= (\mathcal{U} \cap \text{int}_{\text{sf}}(\mathcal{S})) \cup (\mathcal{U} \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}))) \\ &= [(\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \text{int}_{\text{sf}}(\mathcal{Q})] \cup [(\mathcal{U} \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}))) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}))] \\ &= (\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \mathcal{Q} \in \wp(\mathcal{Q}). \end{aligned} \quad (20)$$

For a general case, let $\mathcal{K} = \mathcal{Q} \cap \mathcal{S}$, and we have

$$\begin{aligned} \mathcal{K} \setminus \text{int}_{\text{sf}}(\mathcal{K}) &= (\mathcal{Q} \cap \mathcal{S}) \cap (\text{int}_{\text{sf}}(\mathcal{Q}) \cap \text{int}_{\text{sf}}(\mathcal{S}))^c \\ &= (\mathcal{Q} \cap \mathcal{S} \cap (\text{int}_{\text{sf}}(\mathcal{Q}))^c) \cup (\mathcal{Q} \cap \mathcal{S} \cap (\text{int}_{\text{sf}}(\mathcal{S}))^c) \\ &= (\mathcal{S} \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}))) \cup (\mathcal{Q} \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}))) \\ &= (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) \end{aligned} \quad (21)$$

$$\begin{aligned} \text{cl}_{\text{sf}}(\mathcal{Q} \setminus \mathcal{K}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) &= \text{cl}_{\text{sf}}(\mathcal{Q} \cap (\mathcal{Q}^c \cup \mathcal{S}^c)) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) \\ &= \text{cl}_{\text{sf}}(\mathcal{Q} \cap \mathcal{Q}^c) \cup (\mathcal{Q} \cap \mathcal{S}^c) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) \\ &= \text{cl}_{\text{sf}}(\mathcal{Q} \setminus \mathcal{S}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) = \emptyset. \end{aligned} \quad (22)$$

Similarly, from $\text{cl}_{\text{sf}}(\mathcal{S} \setminus \mathcal{K}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) = \emptyset$, it follows that $\wp(\mathcal{Q}) = \wp(\mathcal{K}) = \wp(\mathcal{S})$.

Conversely, suppose that $\wp(\mathcal{Q}) = \wp(\mathcal{S})$, and let $p \in \mathcal{S} \setminus \text{int}_{\text{sf}}(\mathcal{S})$.

If $p \notin \mathcal{Q}$ and $\mathcal{S} = \mathcal{U} \cup ((\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \mathcal{Q})$, where $\mathcal{U}$ and $\text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)$ are sf-open sets of $\mathfrak{X}$, then $p \in \mathcal{U} \subseteq \mathcal{S}$ and $p \in \text{int}_{\text{sf}}(\mathcal{S})$ indicate a contradiction.

For $p \in \text{int}_{\text{sf}}(\mathcal{Q})$ and $p \in \mathcal{W} \subseteq \mathcal{Q}$, $\mathcal{W} \in \wp(\{p\})$, we have

$$\begin{aligned} p \in \wp(\mathcal{Q}) \cap \text{int}_{\text{sf}}(\mathcal{W} \cap \mathcal{S}) &= \text{int}_{\text{sf}}(\mathcal{W} \cap \mathcal{S}) \cup [\text{int}_{\text{sf}}(\mathcal{W} \cap \mathcal{S} \cap \mathcal{Q})]_{\mathcal{Q}} \\ &= ([\text{int}_{\text{sf}}(\mathcal{W})]_{\mathcal{W}} \cap [\text{int}_{\text{sf}}(\mathcal{S})]_{\mathcal{Q}}) \cap [\text{int}_{\text{sf}}(\mathcal{Q})]_{\mathcal{Q}}. \end{aligned} \quad (23)$$

Then, $(\mathcal{W} \setminus \text{int}_{\text{sf}}(\mathcal{W})) \in \wp(\{p\})$ such that $p \in (\mathcal{W} \setminus \text{int}_{\text{sf}}(\mathcal{W})) \cap \mathcal{Q} \subseteq \mathcal{S}$.

However, $p \in \mathcal{W} \cap (\mathcal{W} \setminus \text{int}_{\text{sf}}(\mathcal{W})) \subseteq (\mathcal{W} \setminus \text{int}_{\text{sf}}(\mathcal{W})) \cap \mathcal{Q} \subseteq \mathcal{S}$, and it follows that $p \in \text{int}_{\text{sf}}(\mathcal{S})$ is a contradiction.

Let $\mathcal{S} \setminus \text{int}_{\text{sf}}(\mathcal{S}) \subseteq \mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})$; the converse similarly holds. Therefore,

$$\mathcal{S} \setminus \text{int}_{\text{sf}}(\mathcal{S}) = \mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}). \quad (24)$$

Finally, for the proof of $\text{cl}_{\text{sf}}(\mathcal{S} \setminus \mathcal{Q}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) = \text{cl}_{\text{sf}}(\mathcal{Q} \setminus \mathcal{S}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) = \emptyset$, suppose that

$\text{cl}_{\text{sf}}(\mathcal{S} \setminus \mathcal{Q}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q})) \neq \emptyset$ such that $p \in \text{cl}_{\text{sf}}(\mathcal{S} \setminus \mathcal{Q}) \cap (\mathcal{Q} \setminus \text{int}_{\text{sf}}(\mathcal{Q}))$ and $p \in \mathcal{U} \in \wp$. Then, any sf-open set

$$\mathcal{O} = \mathcal{U} \cup ((\mathcal{U} \setminus \text{cl}_{\text{sf}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c)) \cap \mathcal{Q}) \quad (25)$$

in $\wp(\mathcal{Q})$ such that $p \in \mathcal{O}$, and we have $\mathcal{O} \cap (\mathcal{Q} \setminus \mathcal{S}) \neq \emptyset$. However, the $\mathfrak{s}\mathfrak{f}$ -open set $\mathcal{W} \cap \mathcal{S}$ in $\wp(\mathcal{S})$ leads to $(\mathcal{W} \cap \mathcal{S}) \cap (\mathcal{S} \setminus \mathcal{Q}) = \emptyset$. Consequently, $\mathcal{W} \cap \mathcal{S} \neq \mathcal{O}$ for each $\mathcal{O}$ in $\wp(\mathcal{Q})$, which contradicts the assumption $\wp(\mathcal{Q}) = \wp(\mathcal{S})$. Hence,

$$cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{S} \setminus \mathcal{Q}) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = \emptyset. \quad (26)$$

Corollary 3.15. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q}, \mathcal{S} \subseteq \mathfrak{X}$ such that $int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) = int_{\mathfrak{s}\mathfrak{f}}(\mathcal{S}) = \emptyset$. Then, $\wp(\mathcal{Q}) = \wp(\mathcal{S})$ if and only if $\mathcal{Q} = \mathcal{S}$.

Proof. Assume that $\wp(\mathcal{Q}) = \wp(\mathcal{S})$ and $\mathcal{Q} \in \wp(\mathcal{S})$. Then, $\mathcal{Q} = \mathcal{U} \cup (\mathcal{V} \cap \mathcal{Q})$, where $\mathcal{U}$ and $\mathcal{V}$ are $\mathfrak{s}\mathfrak{f}$ -open sets of $\mathfrak{X}$. Thus,

$$int_{\mathfrak{s}\mathfrak{f}}(\mathcal{U}) \cup int_{\mathfrak{s}\mathfrak{f}}((\mathcal{U} \cap cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{U} \cap \mathcal{S} \cap \mathcal{Q}^c) \cap \mathcal{Q}) \cap \mathcal{S}) \subseteq int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) = \emptyset, \quad (27)$$

and hence, $\mathcal{Q} = int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) = \emptyset$. Therefore, $\mathcal{Q} \subseteq \mathcal{S}$, and similarly, $\mathcal{S} \subseteq \mathcal{Q}$.

Conversely, can be obtained easily.

Corollary 3.16. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q}, \mathcal{S} \subseteq \mathfrak{X}$. If $\mathcal{S} = \mathfrak{X} \setminus \mathcal{F}$, where $\mathcal{F}$ be a $\mathfrak{s}\mathfrak{f}$ -closed set of $\mathfrak{X}$ and $\mathcal{F} \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = \emptyset$, then $\wp(\mathcal{Q}) = \wp(\mathcal{S})$.

Proof. The following conditions of Theorem (3.14) can be easily depicted as true:

$$\begin{aligned} \mathcal{S} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{S}) &= \mathcal{S} \cap \mathcal{F}^c \cap (int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) \cap \mathcal{F}^c \\ &= [\mathcal{Q} \cap \mathcal{F}^c] \cap [(int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}))^c \cup \mathcal{F}] \\ &= [\mathcal{Q} \cap \mathcal{F}^c \cap (int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}))^c] \cup [\mathcal{Q} \cap \mathcal{F}^c \cap \mathcal{F}] \\ &= (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) \cap \mathcal{F}^c \\ &= \mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}). \end{aligned} \quad (28)$$

Therefore,

$$\mathcal{Q} \setminus \mathcal{S} \subseteq \mathcal{F} \text{ and } cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q} \setminus \mathcal{S}) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) \subseteq \mathcal{F} \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = \emptyset. \quad (29)$$

Corollary 3.17. Let $(\mathfrak{X}, \wp)$ be a top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q}, \mathcal{S} \subseteq \mathfrak{X}$.

If $\mathcal{S} = \mathfrak{X} \cup \mathcal{O}$, where $\mathcal{O}$ is a $\mathfrak{s}\mathfrak{f}$ -open set of $\mathfrak{X}$ with $cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{O}) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = \emptyset$, then $\wp(\mathcal{Q}) = \wp(\mathcal{S})$.

Proof. As $int_{\mathfrak{s}\mathfrak{f}}(\mathcal{S}) = int_{\mathfrak{s}\mathfrak{f}}(\mathfrak{X}) \cup \mathcal{O}$, we have

$$\begin{aligned} (\mathcal{S} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{S})) &= (\mathfrak{X} \cup \mathcal{O}) \cup (int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) \cup \mathcal{O})^c \\ &= (\mathfrak{X} \cap (int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}))^c \cap \mathcal{O}^c) \cup (\mathcal{O} \cap int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q}) \cap \mathcal{O}^c) \\ &= (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) \cap \mathcal{O}^c \\ &= (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})). \end{aligned} \quad (30)$$

Thus, we can easily establish that $(cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q} \setminus \mathcal{S})) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = \emptyset$, and it follows from

$$\mathcal{S} \setminus \mathcal{Q} \subseteq \mathcal{O} \text{ that } cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q} \setminus \mathcal{S}) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{O}) \cap (\mathcal{Q} \setminus int_{\mathfrak{s}\mathfrak{f}}(\mathcal{Q})) = \emptyset. \quad (31)$$

Example 3.18. Let $\mathfrak{X} = \{a, b\}$ have a top $\wp = \wp_{ind}$. Then, $\wp^{\mathfrak{s}\mathfrak{f}} = \wp_{ind}$. Clearly, $(\mathfrak{X}, \wp)$ is a $\mathfrak{s}\mathfrak{f}$ -regular top- $\mathfrak{s}$, which satisfies a conduct union. Let $\mathcal{Q} = \{a\}$. Then, we have $\wp(\mathcal{Q}) = \{\emptyset, \mathfrak{X}, \mathcal{Q}\}$, which is not a $\mathfrak{s}\mathfrak{f}$ -regular.

Theorem 3.19. Let $(\mathfrak{X}, \wp)$ be a $\mathfrak{s}\mathfrak{f}''$ -regular top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q}$ be a $\mathfrak{s}\mathfrak{f}$ -closed set of $\mathfrak{X}$. Then, $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{s}\mathfrak{f}''$ -regular.

Proof. Assume that $(\mathfrak{X}, \wp)$ is a $\mathfrak{s}\mathfrak{f}$ -regular. Furthermore, let $x \in \mathfrak{X}$ and $\mathcal{F}$ be a $\wp(\mathcal{Q})$ -closed set in $(\mathfrak{X}, \wp(\mathcal{Q}))$ such that $x \notin \mathcal{F}$. Some $\wp(\mathcal{Q})$ -nbhd $_{\mathfrak{s}\mathfrak{f}}(x)$ $\mathcal{U}$ do not contain $\wp(\mathcal{Q})$ - $\mathfrak{s}\mathfrak{f}$ -closed $\wp(\mathcal{Q})$ -nbhd $_{\mathfrak{s}\mathfrak{f}}(x)$. Clearly, $x \in \mathcal{Q}$. Assume that $x \notin \mathcal{Q}$ and $\mathcal{U} = \mathcal{V} \cup (\emptyset \cap \mathcal{Q})$ is a $\wp(\mathcal{Q})$ -nbhd $_{\mathfrak{s}\mathfrak{f}}(x)$ $\mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{V}$ in which $(\mathfrak{X}, \wp)$ is a $\mathfrak{s}\mathfrak{f}$ -regular. According to Proposition (3.9.i), $\wp(\mathcal{Q})$ - $cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{W}) \subseteq cl_{\mathfrak{s}\mathfrak{f}}(\mathcal{W})$. Moreover, $\mathcal{W} \cup (\emptyset \cap \mathcal{Q})$ is a $\wp(\mathcal{Q})$ - $\mathfrak{s}\mathfrak{f}$ -closed $\wp(\mathcal{Q})$ -nbhd $_{\mathfrak{s}\mathfrak{f}}(x)$ such that $\mathcal{W} \cup (\emptyset \cap \mathcal{Q}) \subseteq \mathcal{U}$. Hence, $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{s}\mathfrak{f}$ -regular at x .

Theorem 3.20. Let $(\mathfrak{X}, \wp)$ be a $\mathfrak{sf}''$ -regular top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$. Then, $(\mathfrak{X}, \wp(\mathcal{Q}))$ is not a $\mathfrak{sf}''$ -regular if and only if there exists $\mathfrak{x} \in \mathcal{Q}$ such that for each $\mathfrak{sf}$ -nbhd $\mathcal{O}$ of $\mathfrak{x}$, $\mathcal{O} \cap cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q} \neq \emptyset$.

Proof. Assume that $(\mathfrak{X}, \wp)$ is not a $\mathfrak{sf}''$ -regular. Then, there exists $\mathfrak{x} \in \mathfrak{X}$ in which $(\mathfrak{X}, \wp(\mathcal{Q}))$ is not a $\mathfrak{sf}''$ -regular. Furthermore, some $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$ $\mathcal{U}$ do not have a $\wp(\mathcal{Q})$ $\mathfrak{sf}$ -closed $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$. Clearly, $\mathfrak{x} \in \mathcal{Q}$. Assume that $\mathfrak{x} \notin \mathcal{Q}$ and

$\mathcal{U} = \mathcal{V} \cup (\mathcal{O} \cap \mathcal{Q})$ are $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$ $\mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{V}$, where $(\mathfrak{X}, \wp)$ is a $\mathfrak{sf}$ -regular. According to Proposition (3.9.i), $\wp(\mathcal{Q})cl_{\mathfrak{sf}}(\mathcal{W}) \subseteq cl_{\mathfrak{sf}}(\mathcal{W})$. Then, $\mathcal{W} \cup (\mathcal{O} \cap \mathcal{Q})$ is a $\wp(\mathcal{Q})$ $\mathfrak{sf}$ -closed $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$ such that $\mathcal{W} \cup (\mathcal{O} \cap \mathcal{Q}) \subseteq \mathcal{U}$. Thus, $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{sf}''$ -regular at $\mathfrak{x}$.

Furthermore, if some $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$ $\mathcal{B}$ such that $\mathcal{B} \cap (cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q}) = \emptyset$, then $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{sf}''$ -regular at $\mathfrak{x}$.

Let $\mathcal{U} = \mathcal{O} \cup (\mathcal{O} \cap \mathcal{Q})$ be a $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$, and let $\mathcal{V}$ be a $\mathfrak{sf}$ -closed $\mathfrak{sf}$ -nbhd of $\mathfrak{x}$ such that $\mathcal{V} \subseteq \mathcal{B} \cap \mathcal{O}$.

According to Proposition (3.9.iii), $\wp(\mathcal{Q})cl_{\mathfrak{sf}}(\mathcal{V} \cap \mathcal{Q}) = cl_{\mathfrak{sf}}(\mathcal{V} \cap \mathcal{Q}) \subseteq \mathcal{V} \cap cl_{\mathfrak{sf}}(\mathcal{Q})$. Furthermore, $\mathcal{V}$ is $\mathfrak{sf}$ -closed. However, $\mathcal{V} \cap cl_{\mathfrak{sf}}(\mathcal{Q}) = \mathcal{V} \cap \mathcal{Q}$ because $\mathcal{V} \subseteq \mathcal{B} \cap \mathcal{O}$ and $\mathcal{B} \cap cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q} = \emptyset$. Consequently, we can obtain

$\wp(\mathcal{Q})cl_{\mathfrak{sf}}(\mathcal{V} \cap \mathcal{Q}) \subseteq \mathcal{V} \cap \mathcal{Q} \subseteq (\mathcal{B} \cap \mathcal{O}) \cap \mathcal{Q} \subseteq \mathcal{U}$. Thus, $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{sf}''$ -regular at $\mathfrak{x}$, which represents a contradiction.

Conversely, assume that a condition holds. Let $\mathcal{U}$ be a $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$ such that $\mathcal{U} \subseteq \mathcal{Q}$. Assume also that there exists $\mathcal{M}$ as a $\wp(\mathcal{Q})$ $\mathfrak{sf}$ -closed $\wp(\mathcal{Q})nbhd_{\mathfrak{sf}}(\mathfrak{x})$ such that $\mathcal{M} \subseteq \mathcal{U}$. Then, $\mathfrak{x} \in \mathcal{W} \cap \mathcal{Q} \subseteq \mathcal{M}$ for some $\mathfrak{sf}$ -open set $\mathcal{W}$ of $\mathfrak{X}$, and $y \in \mathcal{W} \cap (cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q})$ based on the hypothesis.

$$\text{Hence, } y \in (cl_{\mathfrak{sf}}(\mathcal{W} \cap \mathcal{Q}))^c \subseteq cl_{\mathfrak{sf}}(\mathcal{M}) = \wp(\mathcal{Q})cl_{\mathfrak{sf}}(\mathcal{M}). \quad (32)$$

According to Proposition (3.9.iii), a contradiction exists because $\wp(\mathcal{Q})cl_{\mathfrak{sf}}(\mathcal{M}) = \mathcal{M} \subseteq \mathcal{Q}$.

Theorem 3.21. Let $(\mathfrak{X}, \wp)$ be a $\mathfrak{sf}''$ -regular top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$. Then, $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{sf}''$ -regular if and only if $cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q}$ is a $\mathfrak{sf}$ -closed set of $\mathfrak{X}$.

Proof. For each $\mathfrak{x} \in \mathcal{Q} \cap b_{\mathfrak{sf}}(\mathcal{Q})$, there exists a $\mathfrak{sf}$ -nbhd $\mathcal{W}$ of $\mathfrak{x}$ such that $\mathcal{W} \cap b_{\mathfrak{sf}}(\mathcal{Q}) \subseteq \mathcal{Q} \cap b_{\mathfrak{sf}}(\mathcal{Q})$, which follows immediately Theorems (3.19) and (3.20).

Theorem 3.22. Let $(\mathfrak{X}, \wp)$ be a $\mathfrak{sf}''$ -normal top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q}$ be a $\mathfrak{sf}$ -dens subset of $\mathfrak{X}$. If $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{sf}''$ -regular and $(\mathcal{Q}^c, \wp_{\mathcal{Q}^c})$ is a $\mathfrak{sf}''$ -normal subspace, then $(\mathcal{Q}^c, \wp_{\mathcal{Q}^c}(\mathcal{Q}))$ is a $\mathfrak{sf}''$ -normal.

Proof.

$$\text{Let } \mathcal{Q} \text{ be a } \mathfrak{sf}\text{-dens subset of } \mathfrak{X}. \text{ Then, } cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q} = \mathcal{Q}^c. \quad (33)$$

According to Theorem (3.20), $\mathcal{Q}^c$ is a $\mathfrak{sf}$ -closed set of $\mathfrak{X}$. Moreover, according to Theorems (2.13.ii) and (2.19), we have $(\mathcal{Q}^c, \wp_{\mathcal{Q}^c}(\mathcal{Q}))$, which is a $\mathfrak{sf}''$ -normal.

Theorem 3.23. Let $(\mathfrak{X}, \wp)$ be a $\mathfrak{sf}$ -paracompact top- $\mathfrak{s}$ satisfying a conduct union, and let $\mathcal{Q} \subseteq \mathfrak{X}$. If $\mathcal{Q}^c$ is a $\mathfrak{sf}$ -paracompact subspace of $\mathfrak{X}$, then $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{sf}$ -paracompact.

Proof. Assume that $(\mathfrak{X}, \wp)$ is a $\mathfrak{sf}$ -paracompact. Let $\mathfrak{A}$ be a $\wp(\mathcal{Q})$ $\mathfrak{sf}$ -open cover of $\mathfrak{X}$. Moreover, assume that for each $\mathfrak{x} \in cl_{\mathfrak{sf}}(\mathcal{Q})$, there exists $\mathcal{U} \in \mathfrak{A} \cap \wp$ such that $\mathfrak{x} \in \mathcal{U}$ and $\mathcal{U} \cap \mathcal{Q} = \emptyset$. For each $a \in \mathcal{Q}$, there exists $\mathcal{U} \in \wp$ such that $a \in \mathcal{U} \cap \mathcal{Q}$. Furthermore, for each $y \in cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q}$, there exists $\mathcal{V} \in \mathfrak{A} \cap \wp$ such that $y \in \mathcal{V}$.

$$\text{Let } \mathfrak{B} = \{\mathcal{U} \in \mathfrak{A} : \mathcal{U} \in \wp \text{ and } \mathcal{U} \cap (cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q}) = \emptyset\}. \quad (34)$$

According to Theorem (3.20), as $cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q}$ is a $\mathfrak{sf}$ -closed set of $\mathfrak{X}$, there exists a $\mathfrak{sf}$ -locally finite (and a $\wp(\mathcal{Q})$ $\mathfrak{sf}$ -locally finite) family $\mathfrak{B}'$ of $\mathfrak{sf}$ -open subsets of $\mathfrak{X}$, which $\mathfrak{sf}$ -covers $b_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q}$ and refines $\mathfrak{B}$.

Clearly, $\mathfrak{B} \cup (\mathfrak{X} \setminus cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q})$ is a $\mathfrak{sf}$ -open cover of $\mathfrak{X}$. According to Definition (2.21), $\mathfrak{B} \cup (\mathfrak{X} \setminus cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q})$ is $\mathfrak{sf}$ -locally finite, which is a $\mathfrak{sf}$ -open refinement $\mathfrak{B}_*$. Assume that

$$\mathfrak{B}' = \{\mathcal{W} \in \mathfrak{B}_* : \mathcal{W} \cap b_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q} \neq \emptyset\}, \quad (35)$$

Let $\mathcal{W}' = \bigcup \mathfrak{B}'$. As $\mathcal{Q} \setminus \mathcal{W}' = (\mathcal{Q} \cup b_{\mathfrak{sf}}(\mathcal{Q})) \setminus \mathcal{W}' = cl_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{W}'$ is a $\mathfrak{sf}$ -closed subset of $\mathfrak{X}$, where $b_{\mathfrak{sf}}(\mathcal{Q}) \setminus \mathcal{Q} \subseteq \mathcal{W}'$, we have

$$\mathfrak{B} = \{\mathcal{V} \in \wp : \mathcal{V} \cap \mathcal{Q} \in \mathfrak{A} \text{ and } \mathcal{V} \cap (\mathcal{Q} \setminus \mathcal{W}') \neq \emptyset\}. \quad (36)$$

Then, we can also identify a $\mathfrak{sf}$ -locally finite family $\mathfrak{B}_*$ of $\mathfrak{sf}$ -open sets of $\mathfrak{X}$, which covers $\mathcal{Q} \setminus \mathcal{W}'$ and refines $\mathfrak{B}$. Hence, $\mathfrak{B}' = \{\mathcal{V} \cap \mathcal{Q} : \mathcal{V} \in \mathfrak{B}_*\}$ is a $\wp(\mathcal{Q})$ $\mathfrak{sf}$ -locally finite family of $\wp(\mathcal{Q})$ $\mathfrak{sf}$ -open sets of $\mathfrak{X}$, which refines $\{\mathcal{V} \cap \mathcal{Q} \in \mathfrak{A} : \mathcal{V} \cap (\mathcal{Q} \setminus \mathcal{W}') \neq \emptyset\}$.

Finally,

$$\text{let } \mathfrak{G} = \{\mathcal{U} \in \mathfrak{U} : \mathcal{U} \in \wp, \mathcal{U} \cap (\mathfrak{X} \setminus (\mathcal{W}' \cup \mathcal{Q}) \neq \emptyset \text{ and } \mathcal{U} \cap \mathcal{Q} = \emptyset\}. \quad (37)$$

Clearly,

$c_{\mathfrak{sf}}(\mathcal{Q}) \cup \mathcal{W}' \subseteq \mathcal{W}' \cup \mathcal{Q}$, and hence, $\mathfrak{G}$ covers $\mathfrak{X} \setminus (\mathcal{W}' \cup \mathcal{Q})$. As $\mathfrak{X} \setminus (\mathcal{W}' \cup \mathcal{Q})$ is $\mathfrak{sf}$ -closed in $(\mathcal{Q}^c, \wp_{\mathcal{Q}^c})$, where

$$\mathfrak{X} \setminus (\mathcal{W}' \cup \mathcal{Q}) = \mathcal{Q}^c \setminus \mathcal{W}', \quad (38)$$

then there exists a $\mathfrak{sf}$ -locally finite family $\mathfrak{G}'$ of $\mathfrak{sf}$ -open sets of $(\mathcal{Q}^c, \wp_{\mathcal{Q}^c})$, which covers $\mathfrak{X} \setminus (\mathcal{W}' \cup \mathcal{Q})$ and refines $\mathfrak{G}$. As $\mathcal{Q}^c$ is a $\wp(\mathcal{Q})$ - $\mathfrak{sf}$ -closed set of $\mathfrak{X}$, then $\mathfrak{G}'$ is also $\wp(\mathcal{Q})$ - $\mathfrak{sf}$ -locally finite. Clearly, $\mathfrak{U}' = \mathfrak{B}' \cup \mathfrak{W}' \cup \mathfrak{G}'$ is a $\wp(\mathcal{Q})$ - $\mathfrak{sf}$ -locally finite $\wp(\mathcal{Q})$ - $\mathfrak{sf}$ -open refinement of $\mathfrak{U}$. Thus $(\mathfrak{X}, \wp(\mathcal{Q}))$ is a $\mathfrak{sf}$ -paracompact.

4. CONCLUSION

In this work, we summarized the basic concepts of $\mathfrak{sf}$ -open sets and presented some theoretical studies of the relation between a collection of $\mathfrak{sf}$ -open sets in topological spaces and a new structure of topological space that is a generalization of a topological space and a family of $\mathfrak{sf}$ -open sets, such as the $\mathfrak{sf}$ -simple extension space. Specifically, we studied the $\mathfrak{sf}$ -simple extension space in which a conduct union is a necessary condition.

This work can be extended to certain properties of topological spaces, such as $\mathfrak{sf}''$ -regularity, $\mathfrak{sf}''$ -normality, and $\mathfrak{sf}$ -paracompactness, by using the $\mathfrak{sf}$ -simple extension of topological spaces as a tool. Our proposed approach served as the basis for defining and studying the properties of the abovementioned concept.

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6. CONFLICTS OF INTEREST

The author declares no conflict of interest.

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