Properties of Chmielinski-orthogonality by using Kadets-Klee property

Saied A. Jhonny^{1,*}, Buthainah A. A. Ahmed¹

¹Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

*Corresponding Author: Saied A. Jhonny

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ABSTRACT: This paper aims to study the approximate orthogonality (Chmielinski orthogonality) of Birkhoff–James techniques in real Banach space $(X, \|\cdot\|)$ —even if the concepts symbolised by $\perp_{BJC}^\varepsilon$ orthogonality show no ambiguity—and provide some new geometric characterisations that serve as the basis of our main definitions. This paper also explores the relation between two types of $\perp_{BJC}^\varepsilon$ orthogonalities, namely, $\perp_{BJC}^\varepsilon$ orthogonality in a real Banach space $(X, \|\cdot\|)$ and $\perp_{BJC}^\varepsilon$ orthogonality in the space of the bounded linear operator $\mathcal{B}(X, Y)$. We obtain complete characterisations of these two $\perp_{BJC}^\varepsilon$ orthogonalities in some types of Banach spaces, such as strictly convex, smooth and reflexive spaces. This study provides different results about the concept symmetry of Chmielinski orthogonality for a compact linear operator on a reflexive, strictly convex Banach space having a Kadets–Klee property by exploring new generalised results with Birkhoff–James orthogonality in the space of bounded linear operators. We also obtain a smooth compact linear operator with a spectral value that is defined on a reflexive, strictly convex Banach space having a Kadets–Klee property that is either having zero nullity or not $\perp_{BJC}^\varepsilon$ -right-symmetric.

Keywords: BirkhoffJames orthogonality; Chmielinski orthogonality; symmetry of Chmielinski orthogonality; KadetsKlee property; norm attainment set.

1. INTRODUCTION

The approximation of BirkhoffJames orthogonality in normed spaces has attracted the attention of mathematicians since its introduction by Chmielinski, a founder of functional analysis, in 2005 [1]. Following its introduction, BirkhoffJames orthogonality has become a popular research topic. This research field has grown so vast and even began to significantly intersect with every branch of analysis. Approximating the BirkhoffJames orthogonality also has an increasingly important role in many branches of applied sciences and engineering. This study deals with one of the central approximation theories used in normed spaces.

The Chmielinski-orthogonality introduced by Chmielinski in [1] is the generalised form of the BirkhoffJames orthogonality proposed by James in [2] and [3]. The properties of the BirkhoffJames orthogonality has been widely studied in the literature [4–11]. Chmielinski orthogonality has a key role in investigating the geometry of a Banach space. Previous studies [12–14] have also elaborated how certain notions, such as reflexivity, strict convexity and smoothness, of a Banach space can be studied using Chmielinski orthogonality.

The KadetsKlee property of a Banach space has many geometrical properties, including its strict convexity, which has been related to analytic concepts. A fascinating interplay can also be observed amongst analytic, geometric and operator theoretic aspects.

Several notations and terminologies are used throughout this paper. In the sequel, $(\mathbb{X}, \|\cdot\|)$ will always, $\mathbb{X}$ be real Banach space with norm $\|\cdot\|$ and $\mathbb{X}^*$ its dual. We denote $\mathbb{B}_{\mathbb{X}}$ ($\mathbb{S}_{\mathbb{X}}$, resp.) the closed unit ball (the unit sphere, resp.) of $\mathbb{X}$ and $\mathbb{S}_{\mathbb{X}^*}$ is the unit sphere of $\mathbb{X}^*$.

We use $J(x)$ to denote the non-empty set of all norm one supporting functionals of $\mathbb{B}_{\mathbb{X}}$ at x :

$$J(x) = \{\psi \in \mathbb{S}_{\mathbb{X}^*} : \psi(x) = \|x\|\} \quad (1)$$

A normed space $(\mathbb{X}, \|\cdot\|)$ is said to be have a KadetsKlee property. For any sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{X}$ with $x_n \rightarrow x$ and $\|x_n\| \rightarrow \|x\|$, we have $x_n \rightarrow x$ as $n \rightarrow \infty$.

Let $\mathcal{B}(\mathbb{X}, \mathbb{Y})$ ($\mathcal{K}(\mathbb{X}, \mathbb{Y})$) denote the Banach space of a bounded (compact) linear operators from $(\mathbb{X}, \|\cdot\|)$ to $(\mathbb{Y}, \|\cdot\|)$. We write $\mathcal{B}(\mathbb{X}, \mathbb{Y}) = \mathcal{B}(\mathbb{X})$ ($\mathcal{K}(\mathbb{X}, \mathbb{Y}) = \mathcal{K}(\mathbb{X})$) if $\mathbb{X} = \mathbb{Y}$. A sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{S}_{\mathbb{X}}$ is said to be a norming sequence for $\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$, if $\|\mathcal{T}(x_n)\| \rightarrow \|\mathcal{T}\|$.

We recall that $\mathcal{T} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ attains norm at x in $\mathbb{S}_{\mathbb{X}}$ if $\|\mathcal{T}(x)\| = \|\mathcal{T}\|$. Let $M_{\mathcal{T}}$ denote the set of all vectors x in $\mathbb{S}_{\mathbb{X}}$ at which T attains norm [15], that is,

$$M_{\mathcal{T}} = \{x \in \mathbb{S}_{\mathbb{X}} : \|\mathcal{T}(x)\| = \|\mathcal{T}\|\} \quad (2)$$

A normed space $(\mathbb{X}, \|\cdot\|)$ is said to be smooth if for any vector x in $\mathbb{S}_{\mathbb{X}}$, there exists a unique hyperplane $\mathbb{H}$ (subspace of $\mathbb{X}$ having co-dimension one) supporting $\mathbb{B}_{\mathbb{X}}$ at x [16].

Recall that x in $\mathbb{B}_{\mathbb{X}}$ is an extreme point of $\mathbb{B}_{\mathbb{X}}$ if for any y and z in $\mathbb{B}_{\mathbb{X}}$, $\eta \in (0, 1)$ such that $x = (1 - \eta)y + \eta z$ implies that $x = y = z$.

A Banach space $(\mathbb{X}, \|\cdot\|)$ is said to be strictly convex if and only if for any x , $\mathbb{S}_{\mathbb{X}}$ is an extreme point of $\mathbb{B}_{\mathbb{X}}$ or if for any x and y , $\mathbb{S}_{\mathbb{X}}$ implies that $\|(1 - \eta)x + \eta y\| < 1$ for all $\eta \in (0, 1)$ with $x \neq y$ [17].

For any two vectors x and y in a normed linear space $(\mathbb{X}, \|\cdot\|)$, we have

i. x is orthogonal to y in the sense of BirkhoffJames (for brief $\perp_{BJ}$ orthogonality) and is written as $x \perp_{BJ} y$ if the following property holds true:

$$\|x + \mu y\| \geq \|x\| \text{ for all } \mu \in \mathbb{R} \quad (3)$$

ii. x is orthogonal to y in the sense of Chmielinski (for brief $\perp_{BJC}^{\varepsilon}$ orthogonality) and is written as $x \perp_{BJC}^{\varepsilon} y$ if the following property holds true:

$$\|x + \mu y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\mu y\| \text{ for all } \mu \in \mathbb{R} \quad (4)$$

Otherwise, x is not $\perp_{BJC}^{\varepsilon}$ -orthogonal to y and is written as $x \not\perp_{BJC}^{\varepsilon} y$. The relation between them and several properties of these concepts can be found in [1].

One equivalence definition of Chmielinski orthogonality that allows us to prove the theorems in this paper states that $x \perp_{BJC}^{\varepsilon} y$ if and only if there exists $\psi \in J(x)$ with $|\psi(y)| \leq \varepsilon\|y\|$ [12].

The set $M_{\mathcal{T}}$ has an important role in characterising Chmielinski orthogonality for bounded linear operators on Banach spaces.

In 2020 and modeled on the concept of Chmielinski J. Saied J. and Buthainah A. A., [18] presented some basic results using the following notion:

$$x^{\perp_{BJC}^{\varepsilon}} = \{y \in \mathbb{X} : x \perp_{BJC}^{\varepsilon} y\} \quad (5)$$

where $x^{\perp_{BJC}^{\varepsilon}}$ is the $\perp_{BJC}^{\varepsilon}$ -orthogonal complement of x , and set $\perp_{BJC}^{\varepsilon} x$ in [19] is defined as

$$\perp_{BJC}^{\varepsilon} x = \{y \in \mathbb{X} : y \perp_{BJC}^{\varepsilon} x\} \quad (6)$$

Many researchers have recently examined the symmetricity of the BirkhoffJames orthogonality for a bounded linear operator on a Banach space [20–23].

Saied J. and Buthainah A. A. [18] introduced a complete characterisation for set M_T in a Banach space by using Chmielinski orthogonality techniques. In the same year, Saied J. and Buthainah A. A. [19] examined the symmetricity of Chmielinski orthogonality for vector x in $\mathbb{S}_X$ of a real Banach space and its relationship with the corresponding symmetricity in $J(x)$. In 2021, Saied J. and Buthainah A. A. [24] introduced a complete characterisation for $M_{\mathcal{T}}$ in a smooth Banach space by using Chmielinski orthogonality techniques. In the same year, Saied J. and Buthainah A. A. [25] introduced two notions of points in the geometry of Banach spaces, namely, $\mathcal{C}_{\mathcal{B}}\mathcal{B}$ and $w\mathcal{C}_{\mathcal{B}}\mathcal{B}$, by using the concept of Chmielinski orthogonality and investigated the extreme contraction of a rank one bounded linear operator on a smooth and reflexive Banach space with a KadetsKlee property based on geometric concepts.

The rest of this paper is organised as follows. Section 2 presents those results about the approximate Birkhoff James orthogonality that are either used heavily in the following section or are too fundamental to be left out. Section 3 presents the results obtained based on the symmetric approximate of BirkhoffJames orthogonality in a Banach space with a KadetsKlee property.

2. PRELIMINARIES AND SET BACKGROUND MATERIAL

This section summarises some fundamental theorems and important conclusions regarding Chmielinski orthogonality in Banach spaces.

Proposition 2.1. [18] *Let x and y be any two vectors in a normed space $(X, \|\cdot\|)$. The following statements are satisfied:*
i. If $x \perp_{BJC}^{\varepsilon} y$, then $y \in x^{+(\varepsilon)}$ and $y \in x^{-(\varepsilon)}$, which suggest that

$$x^{\perp}_{BJC}^{\varepsilon} = x^{+(\varepsilon)} \cap x^{-(\varepsilon)} \quad (7)$$

ii. If $y \in x^{+(\varepsilon)} \left(y \in x^{-(\varepsilon)} \right)$, then $\delta y \in (\eta x)^{+(\varepsilon)} \left(\delta y \in (\eta x)^{-(\varepsilon)} \right)$ for all $\delta, \eta > 0$.

iii. If $y \in x^{+(\varepsilon)} \left(y \in x^{-(\varepsilon)} \right)$, then $-y \in x^{-(\varepsilon)} \left(-y \in x^{+(\varepsilon)} \right)$ and $y \in (-x)^{-(\varepsilon)} \left(y \in (-x)^{+(\varepsilon)} \right)$.

Theorem 2.2. [24] *For any two Banach spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ with $\mathcal{T} \in \mathcal{B}(X, Y)$, if $x \in M_{\mathcal{T}}$ and $y = \mathcal{T}(x)$, then there exists the closed hyperspaces $\mathbb{H}_x$ and $\mathbb{H}_y$ of X and Y , respectively, such that $\mathbb{H}_x \subseteq x^{\perp}_{BJC}^{\varepsilon}$ and $\mathbb{H}_y \subseteq y^{\perp}_{BJC}^{\varepsilon}$ with $\mathcal{T}(\mathbb{H}_x) \subseteq \mathbb{H}_y$.*

Theorem 2.3. [24] *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two smooth Banach spaces such that $T \in \mathcal{B}(X, Y)$. If $x \in M_{\mathcal{T}}$, then $\mathcal{T} \left(x^{\perp}_{BJC}^{\varepsilon} \right) \subseteq (\mathcal{T}(x))^{\perp}_{BJC}^{\varepsilon}$.*

Theorem 2.4. [9] *Let $(X, \|\cdot\|)$ be a reflexive Banach space with a KadetsKlee property. Then, for $\psi \in X^*$ we have $M_{\psi} = \pm D$, where D is a compact and connected subset of $\mathbb{S}_X$.*

Theorem 2.5. [13] *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two Banach spaces, where $(X, \|\cdot\|)$ is a reflexive space and $\mathcal{T}, \mathcal{A} \in \mathcal{K}(X, Y)$. Then, $\mathcal{T} \perp_{BJC}^{\varepsilon} \mathcal{A}$ if and only if there exists $x, y \in M_{\mathcal{T}}$ such that*

$$\|\mathcal{T}(x) + \mu \mathcal{A}(x)\|^2 \geq \|\mathcal{T}\|^2 - 2\varepsilon \|\mathcal{T}\| \|\mu \mathcal{A}\|; \text{ for all } \mu \geq 0 \quad (8)$$

$$\|\mathcal{T}(x) + \mu_{\mathcal{C}} \mathcal{A}(x)\|^2 \geq \|\mathcal{T}\|^2 - 2\varepsilon \|\mathcal{T}\| \|\mu_{\mathcal{C}} \mathcal{A}\|; \text{ for all } \mu \leq 0 \quad (9)$$

Theorem 2.6. [13] *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two Banach spaces, where $(X, \|\cdot\|)$ is a reflexive space and $\mathcal{T}, \mathcal{A} \in \mathcal{K}(X, Y)$. Meanwhile, $M_{\mathcal{T}} = D \cup (-D)$, where D is a non-empty compact connected subset of $\mathbb{S}_X$.*

Then, $\mathcal{T} \perp_{BJC}^{\varepsilon} \mathcal{A}$ if and only if there exists $x \in M_{\mathcal{T}}$ such that

$$\|\mathcal{T}(x) + \mu_{\mathcal{C}} \mathcal{A}(x)\|^2 \geq \|\mathcal{T}\|^2 - 2\varepsilon \|\mathcal{T}\| \|\mu_{\mathcal{C}} \mathcal{A}\| \quad (10)$$

Moreover, if $M_{\mathcal{T}} \subseteq M_{\mathcal{A}}$, then $\mathcal{T} \perp_{BJC}^{\varepsilon} \mathcal{A}$ if and only if $\mathcal{T}(x) \perp_{BJC}^{\varepsilon} \mathcal{A}(x)$.

Theorem 2.7. [17] *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed spaces with $\mathcal{T} \in \mathcal{B}(X, Y)$. Then, for any $\mathcal{A} \in \mathcal{B}(X, Y)$, $\mathcal{T} \perp_{BJC}^{\varepsilon} \mathcal{A}$ if and only if either of the conditions in (i) or (ii) holds:*

i. There exists a norming sequence $(x_n)_{n \in \mathbb{N}}$ for $\mathcal{T}$

$$\text{and } \lim_{n \rightarrow \infty} \|\mathcal{A}(x_n)\| \leq \varepsilon \|\mathcal{A}\|.$$

ii. There exist two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $\mathbb{S}_{\mathbb{X}}$ and two sequences $(\eta_n)_{n \in \mathbb{N}}$ and $(\delta_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^+$ such that
 (a) $\eta_n \rightarrow 0, \delta_n \rightarrow 0, \|\mathcal{T}(x_n)\| \rightarrow \|\mathcal{T}\|, \|\mathcal{T}(y_n)\| \rightarrow \|\mathcal{T}\|$ as $n \rightarrow \infty$.
 (b) $\|\mathcal{T}(x_n) + \mu_{\mathcal{C}} \mathcal{A}(x_n)\|^2 \geq$

$$(1 - \eta_n^2) \|\mathcal{T}(x_n)\|^2 - 2\varepsilon \sqrt{1 - \eta_n^2} \|\mathcal{T}(x_n)\| \|\mu_{\mathcal{A}}\| \text{ for all } \mu \geq 0 \quad (11)$$

$$(c) \|\mathcal{T}(y_n) + \mu_{\mathcal{D}} \mathcal{A}(y_n)\|_{\mathbb{Y}}^2 \geq$$

$$(1 - \delta_n^2) \|\mathcal{T}(y_n)\|^2 - 2\varepsilon \sqrt{1 - \delta_n^2} \|\mathcal{T}(y_n)\| \|\mu_{\mathcal{C}} \mathcal{A}\| \text{ for all } \mu \leq 0 \quad (12)$$

3. MAIN RESULTS

We start with the following simple observation.

Theorem 3.1. Let $(\mathbb{X}, \|\cdot\|)$ be a reflexive Banach space with KadetsKlee property. If $\varphi, \psi \in \mathbb{X}^*$ and $M_{\varphi} \cap \ker \psi \neq \emptyset$, then $\psi \in \varphi^{\perp_{BJC}}$.

Proof. From the reflexivity of $\mathbb{X}$ with KadetsKlee property, by using Theorem (2.4), we have $M_{\varphi} = \pm D$ where D is a compact and connected subset of $\mathbb{S}_{\mathbb{X}}$. Given that $M_{\varphi} \cap \ker \psi \neq \emptyset$, there exists $x \in M_{\varphi}$ with $\psi(x) = 0$. Therefore, $\psi(x) \in (\varphi(x))^{\perp_{BJC}}$. Without loss of generality, we assume that $\|\psi\| = 1$. By using Theorem (2.6), we have $\psi \in \varphi^{\perp_{BJC}}$.

Remark 3.2. Here, we have a problem that is still unresolved in the general case. For any $\mathcal{A} \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$ with $\mathcal{A} \in \mathcal{T}^{\perp_{BJC}}$, then $\mathcal{A}(x) \in (\mathcal{T}(x))^{\perp_{BJC}}$ is not needed for some $x \in M_{\mathcal{T}}$. However, each situation may yield different results.

We present the following example to illustrate the above point.

Example 3.3. Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed space whose the unit sphere $\mathbb{S}_{\mathbb{R}^2}$ is given by the regular hexagon with vertices at $\pm(1, 0), \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. According to the description of $\mathbb{S}_{\mathbb{X}}, \mathbb{X}$ is not strictly convex space. $\mathcal{T}$ is defined as

$$\mathcal{T} = \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}.$$

Then, $\|\mathcal{T}\| = 1$ and $M_{\mathcal{T}} = \left\{ \pm(1, 0), \pm(\frac{1}{2}, \frac{\sqrt{3}}{2}), \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2}) \right\}$.

The linear operator $\mathcal{A}$ is defined by setting $\mathcal{A}(1, 0) = (1, 0)$ and $\mathcal{A}(\frac{1}{2}, \frac{\sqrt{3}}{2}) = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. By using Theorem (2.6), we conclude that $\mathcal{A} \in \mathcal{T}^{\perp_{BJC}}$ but there is no $x \in M_{\mathcal{T}}$ such that $\mathcal{A}(x) \in (\mathcal{T}(x))^{\perp_{BJC}}$.

The following theorem is one of the main tools used to study the BJC -symmetricity of a compact linear operator on a reflexive Banach space with KadetsKlee property.

Theorem 3.4. Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ be two Banach spaces, where $(\mathbb{X}, \|\cdot\|)$ is a reflexive space with Kadets Klee property and $\mathcal{J} \in \mathcal{H}(\mathbb{X}, \mathbb{Y})$. For any $\mathcal{A} \in \mathcal{B}(\mathbb{X}, \mathbb{Y}), \mathcal{A} \in \mathcal{T}^{\perp_{BJC}}$ if and only if there exists $x, y \in M_{\mathcal{T}} \cap \ker \mathcal{A}$ with $\mathcal{A}(x) \in (\mathcal{T}x)^{+(\varepsilon)}$ and $\mathcal{A}(y) \in (\mathcal{T}y)^{-(\varepsilon)}$.

Proof. We initially prove the necessary part. Let $\mathcal{A} \in \mathcal{T}^{\perp_{BJC}}$. We claim that for any norming sequence $(x_n)_{n \in \mathbb{N}}$ for $\mathcal{T}$, there exists $x \in M_{\mathcal{T}}$ such that $x_{n_k} \rightarrow x$ for some subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$. From the reflexivity of $\mathbb{X}, \mathbb{B}_{\mathbb{X}}$ is a weakly compact as in [23]. Therefore, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $x_{n_k} \rightarrow x$ and x in $\mathbb{B}_{\mathbb{X}}$.

From the compactness of $\mathcal{T}$, we have $\mathcal{T}(x_{n_k}) \rightarrow \mathcal{T}(x)$, but $(x_n)_{n \in \mathbb{N}}$ is a norming sequence for $\mathcal{T}$. Then, $\|\mathcal{T}(x_n)\| \rightarrow \|\mathcal{T}\|$. From the uniqueness of a limit point, we have $\|\mathcal{T}(x)\| = \|\mathcal{T}\|$ and $x \in M_{\mathcal{T}}$.

In sum, we have $\|x_{n_k}\| = \|x\| = 1$, that is, $\|x_{n_k}\| \rightarrow \|x\|$. However, $x_{n_k} \rightarrow x$ according to a Kadets-Klee property of $\mathbb{X}$, which requires that $x_{n_k} \rightarrow x$. We present the following proof for our claim.

Given that $\mathcal{A} \in \mathcal{T}^{\perp_{BJC}}$, by using Theorem (2.7), either (i) or (ii) holds.

If (i) holds, then following our claim, we have $x \in M_{\mathcal{T}}$ such that $\mathcal{A}(x) = \Theta$ and $\mathcal{A}(x) \in (\mathcal{T}(x))^{\perp_{BJC}^{\varepsilon}}$.
If (ii) holds, then

$$\|\mathcal{T}(x_n) + \mu_{\mathcal{A}}(x_n)\|^2 \geq (1 - \eta_n^2) \|\mathcal{T}(x_n)\|^2 - 2\varepsilon \sqrt{1 - \eta_n^2} \|\mathcal{T}(x_n)\| \|\mu_{\mathcal{A}}\| \quad (13)$$

for all $\mu \geq 0$ and $n \in \mathbb{N}$.

Following our claim, we find $x \in M_{\mathcal{T}}$ such that

$$\|\mathcal{T}(x_n) + \mu_{\mathcal{A}}(x_n)\|^2 \geq \|\mathcal{T}\|^2 - 2\varepsilon \|\mathcal{T}\| \|\mu_{\mathcal{A}}\| \text{ for all } \mu \geq 0 \quad (14)$$

that is, $\mathcal{A}(x) \in (\mathcal{T}(x))^{+(\varepsilon)}$. Similarly, we obtain $y \in M_{\mathcal{T}}$ such that $\mathcal{A}(y) \in (\mathcal{T}(y))^{-(\varepsilon)}$ for all $\mu \leq 0$.

For the sufficient first part, suppose that there exists $x, y \in M_{\mathcal{T}}$ such that

$$\mathcal{A}(x) \in (\mathcal{T}x)^{+(\varepsilon)} \text{ and } \mathcal{A}(y) \in (\mathcal{T}y)^{-(\varepsilon)}. \quad (15)$$

Then, for any $\mu \geq 0$, we have

$$\|\mathcal{T} + \mu_{\mathcal{A}}\mathcal{A}\|^2 \geq \|\mathcal{T}\|^2 - 2\varepsilon \|\mathcal{T}\| \|\mu_{\mathcal{A}}\| \quad (16)$$

and for any $\mu \geq 0$, we have

$$\|\mathcal{T} + \mu_{\mathcal{A}}\mathcal{A}\|^2 \geq \|\mathcal{T}\|^2 - 2\varepsilon \|\mathcal{T}\| \|\mu_{\mathcal{A}}\| \quad (17)$$

Therefore, $\mathcal{A} \in \mathcal{T}^{\perp_{BJC}^{\varepsilon}}$.

Theorem 3.5. Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ be two Banach spaces, where $(\mathbb{X}, \|\cdot\|)$ is reflexive and strictly convex. Suppose that $\mathcal{T} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ is $\perp_{BJC}^{\varepsilon}$ -left-symmetric with $\|\mathcal{T}\| = 1$. If $x_0 \in M_{\mathcal{T}}$ and y_0 in $\mathbb{S}_{\mathbb{X}}$ with $y_0 \in \perp_{BJ} x_0$, then $\mathcal{T}(y_0) = \Theta$.

Proof. Assume that $\mathcal{T}(y_0) \neq \Theta$. Given $y_0 \in \perp_{BJC}^{\varepsilon} x_0$, we have $\psi \in \mathcal{J}(y_0)$ with $|\psi(x_0)| \leq \varepsilon$.

Define $\mathcal{S} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ by $\mathcal{S}(x) = \psi(x)\mathcal{T}(y_0)$ for any vector x in $\mathbb{X}$. Given $x_0 \in M_{\mathcal{T}}$, by using Theorem (2.7.i), we have $\mathcal{S}(x_0) \in (\mathcal{T}(x_0))^{\perp_{BJC}^{\varepsilon}}$. Given $x_0 \in M_{\mathcal{T}}$, then by using Theorem (2.6), we have $\mathcal{S} \in \mathcal{J}^{\perp_{BJC}^{\varepsilon}}$. From the $\perp_{BJC}^{\varepsilon}$ -left-symmetry of $\mathcal{T}$, we have $\mathcal{S} \in \perp_{BJC}^{\varepsilon} \mathcal{T}$. From the strict convexity of $\mathcal{X}$, we deduce $M_{\mathcal{S}} = \{\pm y_0\}$.

Therefore, by using Theorem (2.6), we have $\mathcal{S}(y_0) \in \perp_{BJC}^{\varepsilon}(\mathcal{T}(y_0))$, $\mathcal{T}(y_0) \in \perp_{BJC}^{\varepsilon}(\mathcal{S}(y_0))$ and $\mathcal{T}(y_0) = \Theta$, hence indicating a contradiction.

Theorem 3.6. Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ be two Banach spaces, where $(\mathbb{X}, \|\cdot\|)$ is reflexive, strictly convex and have a Kadets-Klee property. Let $\mathcal{T} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ be $\perp_{BJC}^{\varepsilon}$ -left-symmetric with $M_{\mathcal{T}} \neq \emptyset$. If for any $x_0 \in M_{\mathcal{T}}$ with $y_0 \in \perp_{BJ} x_0$ where y_0 is in $\mathbb{S}_{\mathbb{X}}$, then $\mathcal{T}(y_0) = \Theta$.

Proof. The result is obtained by applying theorems (3.4) and (3.5).

We then prove the following theorem, which is needed in our main result.

Theorem 3.7. Let $(\mathbb{X}, \|\cdot\|)$ be a strictly convex Banach space, where u and v are any two vectors in $\mathcal{X}$. Let $v \in \perp_{BJC}^{\varepsilon} u$ and $w = (1-t)u + tv$, where $t \in (0, 1)$. Then, $au \notin (av + bw)^{-(\varepsilon)}$ whenever $ab > 0$.

Proof. For any $t \in (0, 1)$, we have

$$av + bw = av + b(1-t)u + btv = (a+bt)v + b(1-t)u \quad (18)$$

First case: If $a > 0$ and $b > 0$, then $a+bt > 0$ and $b(1-t) > 0$. From the strict convexity of $\mathbb{X}$ and $v \in \perp_{BJC}^{\varepsilon} u$, we have

$$\|(a+bt)v + b(1-t)u\| \geq \|(a+bt)v\| \quad (19)$$

Given

$$\begin{aligned} \|(av + bw) - b(1-t)u\| &= \|(a + bt)v\| \\ &< \|(a + bt)v + b(1-t)u\| = \|av + bw\| \end{aligned} \quad (20)$$

we have $u \notin (av + bw)^{-(\varepsilon)}$. Following Proposition (2.1.ii), we have $au \notin (av + bw)^{-(\varepsilon)}$.

Second case: If $a < 0$ and $b < 0$, then

$$\begin{aligned} \|(av + bw) - b(1-t)u\| &= \|(a + bt)v\| \\ &< \|(a + bt)v + b(1-t)u\| = \|av + bw\| \end{aligned} \quad (21)$$

where $-b(1-t) > 0$.

Therefore, $u \notin (av + bw)^{+(\varepsilon)}$. Following Propositions (2.1.ii) and (2.1.iii), we have $au \notin (av + bw)^{-(\varepsilon)}$.

Using similar argument as in Theorem (3.6), we can prove the following theorem for $\perp_{BJC}^E$ -left-symmetric operators by making an additional assumption of strict convexity on the range space.

Theorem 3.8. Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ be two Banach spaces and strictly convex, where $(\mathbb{X}, \|\cdot\|)$ is reflexive and have a KadetsKlee property. Let $\mathcal{T} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ with $M_{\mathcal{T}} \neq \emptyset$. If $\mathcal{T}$ is a $\perp_{BJC}^E$ -left-symmetric, then $\mathcal{T} = \Theta$.

Proof. Assume that $\mathcal{T}$ is $\perp_{BJC}^E$ -left-symmetric with $\mathcal{T} \neq \Theta$, and let $x_0 \in M_{\mathcal{T}}$. Following Theorem (3.6), we have $\mathcal{T}(y_0) = \Theta$ for any y_0 in $\mathbb{S}_{\mathbb{X}}$ with $y_0 \in \perp_{BJC}^E x_0$.

Given $x_0 \in M_{\mathcal{T}}$, by using Theorem (2.2), we observe a closed hyperspace $\mathbb{H}_{x_0}$ such that $\mathbb{H}_{x_0} \subseteq x_0^{\perp_{BJC}^E}$ and $\mathcal{T}(\mathbb{H}_{x_0}) \subseteq (\mathcal{T}(x_0))^{\perp_{BJC}^E}$. We claim that $\mathcal{T}(\mathbb{H}_{x_0}) = \Theta$. Let $y_0 \in \mathbb{H}_{x_0} \cap \mathbb{S}_{\mathbb{X}}$. Then, there exists a closed hyperspace $\mathbb{H}_{y_0}$ such that $\mathbb{H}_{y_0} \subseteq y_0^{\perp_{BJC}^E}$.

Define a linear operator $\mathcal{S}$ as $\mathcal{S}(\gamma y_0 + h) = \gamma \mathcal{T}(y_0)$, where $\gamma \in \mathbb{R}$ and $h \in \mathbb{H}_{y_0}$. Clearly, $\mathcal{S} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ and $M_{\mathcal{S}} = \{\pm y_0\}$.

Let $x_0 = \delta y_0 + h$. Therefore, $\mathcal{S}(x_0) = \delta \mathcal{T}(y_0)$. Given $y_0 \in x_0^{\perp_{BJC}^E}$ and $\mathcal{T}(y_0) \in (\mathcal{T}(x_0))^{\perp_{BJC}^E}$, we have $\mathcal{S}(x_0) \in (\mathcal{T}(x_0))^{\perp_{BJC}^E}$. By using Theorem (2.6), we have $\mathcal{S} \in \mathcal{T}^{\perp_{BJC}^E}$. From the $\perp_{BJC}^E$ -left-symmetry of $\mathcal{T}$, we have $\mathcal{S} \in \perp_{BJC}^E \mathcal{T}$. Given that $y_0 \in M_{\mathcal{S}}$, by using Theorem (2.6), we have $\mathcal{S}(y_0) \in \perp_{BJC}^E (\mathcal{T}(y_0))$, but $\mathcal{S}(y_0) = \mathcal{T}(y_0)$. Therefore, $\mathcal{T}(y_0) \in \perp_{BJC}^E (\mathcal{T}(y_0))$ and $\mathcal{T}(y_0) = \Theta$. Our claim $\mathcal{T}(\mathbb{H}_{x_0}) = \Theta$ is then established, and we have $\mathcal{T}(h) = \Theta$ for all $h \in \mathbb{H}_{x_0}$.

There exists $\rho \in \mathbb{R}$ such that $(\rho x_0 + h) \in \perp_{BJC}^E x_0$ for all $h \in \mathbb{H}_{x_0}$. By using Theorem (3.6), we have $\mathcal{T}(\rho x_0 + h) = \Theta$, but $\mathcal{T}(x_0) \neq \Theta$. Therefore, $\rho = 0$, which produces $h \in \perp^{\varepsilon \varepsilon JC} x_0$ and $\mathbb{H}_{x_0} \subseteq \perp_{BJC}^E x_0$.

We then prove that $\mathcal{T}(x_0)$ is $\perp_{BJC}^E$ -left-symmetric. Let $u \in \mathcal{T}(x_0)^{\perp_{BJC}^E}$, and define $\mathcal{S} \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ as $\mathcal{S}(\eta x_0 + h) = \eta u$, where $u \in \mathbb{R}$ and $h \in \mathbb{H}_{x_0}$.

Clearly, $M_{\mathcal{S}} = \{\pm x_0\}$. Given $\mathcal{S}(x_0) \in \mathcal{T}(x_0)^{\perp_{BJC}^E}$ and $x_0 \in M_{\mathcal{S}}$, we have $\mathcal{S} \in \mathcal{T}^{\perp_{BJC}^E}$. From the $\perp_{BJC}^E$ -left-symmetry of $\mathcal{T}$, we have $\mathcal{S} \in \perp_{BJC}^E \mathcal{T}$. Therefore, by using Theorem (2.6), we have $\mathcal{S}(x_0) \in \perp^{\varepsilon \varepsilon JC} (\mathcal{T}(x_0))$, that is, $u \in \perp_{BJC}^E (\mathcal{T}(x_0))$.

We then construct $\mathcal{A} \in \mathcal{K}(\mathbb{X})$ such that $\mathcal{A} \in \mathcal{T}^{\perp_{BJC}^E}$ but $\mathcal{A} \notin \perp_{BJC}^E \mathcal{T}$. Let $y_0 \in \mathbb{H}_{x_0} \cap \mathbb{S}_{\mathbb{X}}$. Clearly, $y_0 \in x_0^{\perp_{BJC}^E}$ and $y_0 \in \perp_{BJC}^E x_0$. Then, $\mathbb{H}_{y_0} \subseteq y_0^{\perp_{BJC}^E}$, where $\mathbb{H}_{y_0}$ is a closed subspace of $\mathbb{H}_{x_0}$ having a co-dimension of 1. Let v in $\mathbb{S}_{\mathbb{X}}$ such that $v \in (\mathcal{T}(x_0))^{\perp_{BJC}^E}$. From the $\perp_{BJC}^E$ -left-symmetry of $\mathcal{T}(x_0)$, we have $v \in \perp_{BJC}^E (\mathcal{T}(x_0))$.

We then place $\|x_0 + y_0\| = r$. By using the $\perp_{BJC}^E$ -orthogonality of x_0 and the strict convexity of $\mathbb{X}$, we have $1 \leq r < 2$.

We select $0 < t < 1$, such that $(1-t)(1 + \|\mathcal{T}\|) < \frac{2-r}{1+2r} < 1$.

We then choose $\delta \in (0, 1)$ with $(1-t)(1 + \|\mathcal{T}\|) < \delta < \frac{2-r}{1+2r}$. Let $w = (1-t)\mathcal{T}(x_0) + tv$. Then,

$$\|w - v\| = (1-t)\|\mathcal{T}(x_0) - v\| \leq (1-t)(1 + \|\mathcal{T}\|) < \delta \quad (22)$$

Any z in $\mathbb{X}$ can be written as $z = ax_0 + by_0 + h$, where $a, b \in \mathbb{R}$ and $h \in \mathbb{H}_{y_0}$.

Define $\mathcal{A} \in \mathcal{K}(\mathbb{X})$ as $\mathcal{A}(z) = av + bw$. Given $x_0 \in M_{\mathcal{S}}$ and $\mathcal{A} \in \mathcal{T}^{\perp_{BJC}^E}$, we have $\mathcal{A}(x_0) \in (\mathcal{T}(x_0))^{\perp_{BJC}^E}$.

We next show that $\mathcal{A} \notin \perp_{BJC}^E \mathcal{T}$.

$$\left\| \mathcal{A} \left(\frac{x_0 + y_0}{\|x_0 + y_0\|} \right) \right\| = \frac{\|v + w\|}{r} = \frac{\|2v + w - v\|}{r} \leq \frac{2\|v\| + \|w - v\|}{r} < \frac{2 + \delta}{r} \quad (23)$$

However, $\frac{2+\delta}{r} > 1 + 2\delta$. Therefore, $\|\mathcal{A}\| \geq 1 + 2\delta$. We can easily observe that $x_0, y_0 \notin M_{\mathcal{A}}$. For each $h \in \mathbb{H}_{y_0}$, we have $h \notin M_{\mathcal{A}}$.

We claim that $z = ax_0 + by_0 + h \notin M_{\mathcal{A}}$ whenever $ab < 0$. Let $z = -ax_0 + by_0 + h$ in $\mathbb{S}_{\mathbb{X}}$, where $a > 0$ and $b > 0$ with $ba > 0$. By using the $\perp_{BJC}^{\varepsilon}$ -orthogonality and strict convexity of $\mathbb{X}$, we have

$$1 = \|-ax_0 + by_0 + h\| > a \text{ and } 1 = \|by_0 + h - ax_0\| \geq b - a$$

Moreover, $|b| = \|by_0\| \leq \|by_0 + h\| = \|z + ax_0\| \leq 2$. Then,

$$\begin{aligned} \|\mathcal{A}(z)\| &= \|-av + bw\| = \|(b-a)v + b(w-v)\| \leq |b-a| + |b|\|w-v\| \\ &< b-a + b\delta \leq 1 + 2\delta < \left\| \mathcal{A} \left(\frac{x_0 + y_0}{\|x_0 + y_0\|} \right) \right\| \end{aligned} \quad (24)$$

We consider $z = -ax_0 + by_0 + h$, where $a > 0, b > 0$ with $b-a \leq 0$. Then,

$$\|\mathcal{A}(z)\| = \|-av + bw\| = \|(b-a)v + b(w-v)\| < |b-a| + |b|\delta < a < 1 \quad (25)$$

which shows that if $z = -ax_0 + by_0 + h$ in $\mathbb{S}_{\mathbb{X}}$, where $a > 0$ and $b > 0$, then $z \notin M_{\mathcal{A}}$. Similarly, given $z = ax_0 - by_0 + h$ in $\mathbb{S}_{\mathbb{X}}$, we can show that $z \notin M_{\mathcal{A}}$. Therefore, if $z = ax_0 + by_0 + h$, then $ab > 0$.

We then claim that $\mathcal{T}(z) \notin (\mathcal{A}(z))^{-(\varepsilon)}$ for all $z \in M_{\mathcal{A}}$.

Let $z = ax_0 + by_0 + h \in M_{\mathcal{A}}$ with $ab > 0$. Then, let $\mathcal{A}(z) = av + bw$ and $\mathcal{T}(z) = a\mathcal{T}(x_0)$. By using Theorem (3.7), we have $T(z) \notin (\mathcal{A}(z))^{-(\varepsilon)}$. By using Theorem (3.4), we have $\mathcal{A} \notin \perp_{BJC}^{\varepsilon} \mathcal{T}$.

The following theorem posits that $\mathcal{T}$ is $\perp_{BJC}^{\varepsilon}$ -right-symmetric when $\mathcal{T} \in \mathcal{K}(\mathbb{X})$ has a spectral value.

Theorem 3.9. Let $(\mathbb{X}, \|\cdot\|)$ be a reflexive, strictly convex and Banach space with a Kadets-Klee property. If $\mathcal{J} \in \mathcal{K}(\mathbb{X})$ is a smooth and $\|\mathcal{T}\|$ is a spectral value of $\mathcal{T}$, then either nullity $\mathcal{T} = \Theta$ or $\mathcal{T}$ is not $\perp_{BJC}^{\varepsilon}$ -right-symmetric.

Proof. Given that $\mathbb{T}$ is compact and smooth on a reflexive Banach space, from [9, Th.4.2], there exists x in $\mathbb{S}_{\mathbb{X}}$ such that $M_{\mathcal{T}} = \{\pm x\}$.

Given that every non-zero spectral value of $\mathcal{T}$ is an eigenvalue of $\mathcal{T}$ with $M_{\mathcal{T}} = \{\pm x\}$, we have $\mathcal{T}(x) = \|\mathcal{T}\|x$ or $\mathcal{T}(x) = -\|\mathcal{T}\|x$.

Assume that nullity $\mathcal{T} \geq 1$. Let u be a vector in $\mathbb{S}_{\mathbb{X}}$ such that $\mathcal{T}(u) = \Theta$. Then, for any $\mu \in \mathbb{R}$, we have $\|\mathcal{J} + \mu\mathcal{T}\|^2 \geq \|(\mathcal{J} + \mu\mathcal{T})(u)\|^2 = \|\mathcal{J}(u)\|^2 = \|\mathcal{J}\|^2$

$$\geq \|\mathcal{J}\|^2 - 2\varepsilon\|\mathcal{J}\|\|\mu\mathcal{T}\| \quad (26)$$

Therefore, $\mathcal{J} \in \perp_{BJC}^{\varepsilon} \mathcal{T}$. Suppose that $\mathcal{J} \in \mathcal{J}^{\perp_{BJC}^{\varepsilon}}$. Given $M_{\mathcal{T}} = \{\pm x\}$, by using Theorem (3.4), we have $\mathcal{J}(x) \in (\mathcal{T}(x))^{\perp_{BJC}^{\varepsilon}}$. Consequently, $x \in (\pm\|\mathcal{T}\|\mathcal{J}(\mathbb{X})x)^{\perp_{BJC}^{\varepsilon}}$, which indicates a contradiction. Therefore, $\mathcal{T}$ is not $\perp_{BJC}^{\varepsilon}$ -right-symmetric.

Remark 3.10. We show that in Theorem (3.9), the strict convexity of $(\mathbb{X}, \|\cdot\|)$ is required. Moreover, in example (3.3), assuming that X is strictly convex would suffice.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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