Spectral CG Algorithm for Solving Fuzzy Nonlinear Equations

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ABSTRACT: The nonlinear conjugate gradient method is an effective technique for solving large-scale minimizations problems, and has a wide range of applications in various fields, such as mathematics, chemistry, physics, engineering and medicine. This study presents a novel spectral conjugate gradient algorithm (non-linear conjugate gradient algorithm), which is derived based on the Hisham–Khalil (KH) and Newton algorithms. Based on pure conjugacy condition The importance of this research lies in finding an appropriate method to solve all types of linear and non-linear fuzzy equations because the Buckley and Qu method is ineffective in solving fuzzy equations. Moreover, the conjugate gradient method does not need a Hessian matrix (second partial derivatives of functions) in the solution. The descent property of the proposed method is shown provided that the step size α_k meets the strong Wolfe conditions. In numerous circumstances, numerical results demonstrate that the proposed technique is more efficient than the Fletcher–Reeves and KH algorithms in solving fuzzy nonlinear equations.

Keywords: algorithms; CG methods; descent; Fuzzy; optimization

1. INTRODUCTION

Iterative approaches for solving nonlinear equations such as

$$F(x) = 0 \quad (1)$$

have received considerable interest in recent years. The concept of fuzzy numbers and the mathematical operations that can be performed on them were first proposed and studied by Zadeh [1]. One of the most common applications of fuzzy number arithmetic is nonlinear equations, the parameters of which are completely or partially represented by fuzzy numbers [2–4]. Buckley and Qu's standard analytical procedures [5–8] are only suitable for the linear and quadratic cases of nonlinear equations and cannot be used to solve the following equations:

- I- $Qy^3 + Vy^2 - Ly = \Theta$
- II- $\Theta e^y - Ky = A$
- III- $\Phi y \csc(y) + \Upsilon y = \Omega$ and
- IV- $\Psi y^5 - \Omega \cot(y) = \Phi$

where $y, Q, V, L, \Theta, K, A, \Phi, \Upsilon, \Omega$, and Ψ are fuzzy numbers. In 2004, S. Abbasbandy and B. Asady used Newton's approach for solving a fuzzy nonlinear equation in [9]. In 2010, Amirah Ramli, Mohd Lazim Abdullah and Mustafa Mamat utilised quasi Newton technique to solve a fuzzy nonlinear equation in [10]. However, this method is not $[H_i]$, computation of

the elements of matrix $[H_i]$ becomes extremely difficult and often impossible, inversion of matrix $[H_i]$ at each step and assessment of amount $[H_i]^{-1} \nabla f_i$ at each step. The strategy is ineffective for challenges with a complex objective function with numerous variables because of these flaws. S. Abbasbandy and A. Jafarian developed the steepest descent approach for solving fuzzy nonlinear equations in [11], which they published in 2004. However, this strategy is weak and inefficient in the majority of applications because the sharpest descent direction is a local feature. In 2021, Hisham and Khalil developed two conjugate gradient (CG) methods for solving fuzzy nonlinear equations in [12, 13]. A disadvantage of CG algorithms is that they are not consistently effective. Therefore, we need to use the novel spectral CG algorithm to find the roots of these equations. This study uses one of the spectral conjugate gradient algorithms because it is efficient, fast in finding roots of equations and also has global convergence. The remainder of this paper is organized as follows into six parts. The first part provides a general introduction, previous studies and the importance of this research. The second part presents the basics of arithmetic operations for fuzzy numbers. The third part provides an explanation of the CG algorithms. The fourth part introduces the proposed SKH- CG algorithm. The fifth part presents the computational results and comparisons. Lastly, the sixth part provides the conclusions, acknowledgements and references.

2. PRELIMINARIES

This section discusses some basic definitions and arithmetic operations for fuzzy numbers. We refer interested readers to [14] for additional information.

Definition 2.1. A fuzzy number is defined as a set $j : \mathbb{R} \longrightarrow I = [0, 1]$ that meets the following conditions [15]:

- a. j denotes a semi-continuous upper boundary;
- b. $j(x) = 0$ outside some range $[r, t]$;
- c. there exist $p, q \in \mathbb{R}$, such that $r \leq p \leq q \leq t$ and
 - i. $j(x)$ is increasing monotonically on $[r, p]$,
 - ii. $j(x)$ is decreasing monotonically on $[q, t]$ and
 - iii. $j(x) = 1, p \leq x \leq q$.

Definition 2.2. $j : \mathbb{R} \longrightarrow I = [0, 1]$ is in parametric form and refers to the pair $(\underline{j}, \bar{j})$ of $\underline{j}(x), \bar{j}(x), 0 \leq \mu \leq 1$ satisfying [15, 16]:

- (1) $\underline{j}(x)$ is a monotonically bounded growing left continuous function;
- (2) $\bar{j}(x)$ is a monotonically bounded growing right continuous function and
- (3) $\underline{j}(x) \leq \bar{j}(x)$.

Definition 2.3. A classical fuzzy number h refers to the triangular number $j = (p, r, q)$, which is given in equation (2):

$$j(x) = \begin{cases} \frac{(x-p)}{(r-p)}, & p \leq x \leq r \\ \frac{(x-q)}{(r-q)}, & r \leq x \leq q \end{cases} \quad (2)$$

where $j(x)$ is known as the membership function and $r \neq p, r \neq q$ [9], [17]. This function can be written in the following parameterized form:

$$\begin{aligned} \bar{j}(\mu) &= q + (r-q)\mu \\ \underline{j}(\mu) &= p + (r-p)\mu \end{aligned}$$

$TF(\square)$ is assumed to denote the set of all trapezoidal fuzzy number. The following extension principle can be used to extend the scalar multiplication and addition operations to fuzzy numbers [9].

Let $j = (\underline{j}(\mu), \bar{j}(\mu)), k = (\underline{k}(\mu), \bar{k}(\mu))$ with $w > 0$. The addition $(j + k)$ and multiplication by scalar w are defined as follows:

$$\begin{aligned} (\overline{j+k})(\mu) &= \bar{j}(\mu) + \bar{k}(\mu) \\ (\underline{j+k})(\mu) &= \underline{j}(\mu) + \underline{k}(\mu) \\ (\overline{wj})(\mu) &= w\bar{j}(\mu) \\ (\underline{wj})(\mu) &= w\underline{j}(\mu) \end{aligned}$$

3. CG ALGORITHMS

The form of the nonlinear CG method is presented in equation (3):

$$x_{k+1} = x_k + \alpha_k d_{k+1}, k \geq 1 \quad (3)$$

where x_1 , is an initial point, α_k is a step-length and α_k is a step-size that satisfy the standard Wolfe conditions [17–19] in equations (4) and (5):

$$f(x_k + \alpha_k d_k) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T d_k \quad (4)$$

$$\nabla f(x_k + \alpha_k d_k)^T d_k \geq c_2 \nabla f(x_k)^T d_k \quad (5)$$

or strong Wolfe conditions [20–30] in equations (6) and (7):

$$f(x_k + \alpha_k d_k) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T d_k, \quad (6)$$

$$\left| \nabla f(x_k + \alpha_k d_k)^T d_k \right| \leq c_2 \left| \nabla f(x_k)^T d_k \right|, \quad (7)$$

$$d_{k+1} = \begin{cases} -g_1, & k = 1 \\ -g_{k+1} + \beta_k d_k, & k \geq 1 \end{cases} \quad (8)$$

Different β_{k+1} will determine a different CG method [29]. Some popular formula for β_{k+1} are as follows:

Fletcher and Reeves (FR) [31], Fletcher (CD) [32], Polak and Ribiere (PRP) [33], Hestenes-Stiefel (HS) [34], Dai-Yuan (DY) [35] and Hisham- Khalil (KH) [12], in which β is scalar.

$$\begin{aligned} \beta_k^{FR} &= \frac{\|g_{k+1}\|^2}{\|g_k\|^2} & \beta_k^{CD} &= \frac{\|g_{k+1}\|^2}{-g_k^T d_k} \\ \beta_k^{HS} &= \frac{g_{k+1}^T y_k}{d_k^T y_k} & \beta_k^{PRP} &= \frac{g_{k+1}^T y_k}{\|g_k\|^2} \\ \beta_k^{DY} &= \frac{\|g_{k+1}\|^2}{y_k^T d_k} & \beta_k^{KH} &= \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} \end{aligned}$$

where $g_k = \nabla f(x_k)$, and let $y_k = g_{k+1} - g_k$.

4. PROPOSED SKH-CG ALGORITHM

The search direction for the Hisham-Khalil (KH) CG (KH-CG) method is obtained by $d_1 = g_1$ and

$$d_{k+1} = -g_{k+1} + \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} d_k.$$

Although the methods Fletcher and Reeves (FR) [31], Hisham- Khalil (KH) [12], Dai and Yuan (DY) [35] and Fletcher (CD) [32] methods have strong global convergence properties, these methods have modest practical performance. Meanwhile, the methods of Hestenes and Stiefel (HS) [34] and Polak and Ribiere (PR) [33] methods are not consistently convergent, but they often have good computational properties [36]. To obtain the conjugate gradientCG methods with computational efficiency and good convergence properties, the algorithms are found to avoid failure and improve the performance of classical conjugated gradientCG algorithms. We use the following equation to accelerate the Hisham-Khalil (KH-CG) method.

Let $\gamma_{k+1} = 1 + \mu_{k+1}$

where γ_{k+1} and μ_{k+1} are two parameters. Thereafter,

$$\begin{aligned} d_{k+1} &= -\gamma g_{k+1} + \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} d_k, \\ d_{k+1} &= -(1 + \mu_{k+1}) g_{k+1} + \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} d_k. \end{aligned} \quad (9)$$

We incorporate the second order information to the search direction in equation (9) by assuming that the direction in (9) this equation is parallel to the Newton direction:

$$-G_{k+1}^{-1} g_{k+1} = -g_{k+1} - \mu_{k+1} g_{k+1} + \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} d_k \quad (10)$$

where G_{k+1}^{-1} is the inverse Hessian matrix. Suppose G_{k+1}^{-1} is symmetric ($G_{k+1}^{-1} = (G_{k+1}^{-1})^T$), positive definite and satisfies the quasi-Newton condition:

$$G_{k+1}^{-1}y_k = s_k, \quad (11)$$

where $s_k = x_{k+1} - x_k$ multiply both sides of equation (10) by and considering G_{k+1}^{-1} to be symmetric and positive definite. Then

$$-(G_{k+1}^{-1}g_{k+1})^T g_{k+1} = -y_k^T g_{k+1} - \mu_{k+1}y_k^T g_{k+1} + \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} y_k^T d_k$$

The relation given in equation is used (11) to obtain:

$$-s_k^T g_{k+1} = -y_k^T g_{k+1} - \mu_{k+1}y_k^T g_{k+1} + \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} y_k^T d_k$$

By dividing both sides in the preceding equation by $y_k^T d_k$

$$-\frac{s_k^T g_{k+1}}{y_k^T d_k} = -\beta_{k+1}^{HS} - \mu_{k+1}\beta_{k+1}^{HS} + \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2}, \text{ (Where } \beta_{k+1}^{HS} = \frac{y_k^T g_{k+1}}{y_k^T d_k}, \text{ and } \beta_{k+1}^{KH} = \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2})$$

or

$$\begin{aligned} -\frac{s_k^T g_{k+1}}{y_k^T d_k} &= -\beta_{k+1}^{HS} - \mu_{k+1}\beta_{k+1}^{HS} + \beta_{k+1}^{KH} \\ -\frac{s_k^T g_{k+1}}{y_k^T d_k} &= -(1 + \mu_{k+1})\beta_{k+1}^{HS} + \beta_{k+1}^{KH} \end{aligned}$$

or

$$\begin{aligned} (1 + \mu_{k+1})\beta_{k+1}^{HS} &= \frac{s_k^T g_{k+1}}{y_k^T d_k} + \beta_{k+1}^{KH} \\ \therefore \gamma_{k+1} &= \frac{\beta_{k+1}^{KH}}{\beta_{k+1}^{HS}} + \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}} \end{aligned} \quad (12)$$

This value is used for γ_{k+1} in equation (9):

$$d_{k+1} = -\gamma_{k+1}g_{k+1} + \beta_{k+1}^{KH}d_k \quad (13)$$

We call the algorithm defined in equations (12) and (13) as spectral Hisham-Khalil (SKH-CG) algorithm and we summarise it as the SKH-CG algorithm.

4.1 ALGORITHM (SKH-CG)

Step (1): Initialization : Select $x_1 \in R^n$, $\varepsilon > 0$ is a small positive real value and compute $d_1 = -g_1$, $\alpha_k = \frac{1}{\|g_1\|}$ and $k = 1$.

Step (2): Test for convergence: If $\|g_k\| \leq \varepsilon$ break x_k is optimal solution; else go to Step (3).

Step (3): Line search: Calculate α_k satisfying the strong Wolfe conditions in equations (6) and (7), update the variable $x_{k+1} = x_k + \alpha_k d_{k+1}$ and calculate f_{k+1}, g_{k+1}, y_k and s_k .

Step (4): Direction calculation: Calculate γ_{k+1} from equation (12), if $\gamma_{k+1} \geq 1$ or $\gamma_{k+1} \leq 0$ set $\gamma_{k+1} = 1$ and $d_{k+1} = -\gamma_{k+1}g_{k+1} + \beta_{k+1}^{KH}d_k$ then $d_{k+1} = -\gamma_{k+1}g_{k+1}$ else $d_{k+1} = d$ and $\alpha_{k+1} = \alpha_k \frac{\|d_k\|}{\|d_{k+1}\|}$, $k = k + 1$ go to Step (2).

4.2 DESCENT PROPERTY

This part proves that our algorithm defined in equations (21) and (21) generates descent direction for all iterations according to the following theorem.

THEOREM. Consider the algorithm defined in equation (1), where d_k is computed from equations (21) and (21). Assume that the step size α_k satisfies the strong Wolfe conditions in equations (6) and (7). Thereafter, the search directions d_k generated by the SKH-CG algorithm are descent for all k , provided $y_k^T g_{k+1} > 0$.

Proof. The proof is by indication, for $k = 1, d_1 = -g_1 \rightarrow g_1^T d_1 < 0$

Suppose $g_k^T d_k < 0$ or $g_k^T s_k < 0$ $s_k = \alpha_k d_k$. For $k + 1$, we have:

$$\begin{aligned} d_{k+1} &= -\left(\frac{\beta_{k+1}^{KH}}{\beta_{k+1}^{HS}} + \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1} + \beta_{k+1}^{KH} d_k \\ g_{k+1}^T d_{k+1} &= -\left(\frac{\beta_{k+1}^{KH}}{\beta_{k+1}^{HS}} + \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1}^T g_{k+1} + \beta_{k+1}^{KH} g_{k+1}^T d_k \\ g_{k+1}^T d_{k+1} &= -\left(\frac{\beta_{k+1}^{KH}}{\beta_{k+1}^{HS}} + \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1}^T g_{k+1} + \frac{\beta_{k+1}^{KH}}{\alpha_k} g_{k+1}^T s_k \\ g_{k+1}^T d_{k+1} &= -\left(\frac{\beta_{k+1}^{KH}}{\beta_{k+1}^{HS}} + \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1}^T g_{k+1} + \frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2 g_{k+1}^T s_k} \end{aligned}$$

Divide both sides by $\frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2}$.

$$\begin{aligned} \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &= -\alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} \left(\frac{\beta_{k+1}^{KH}}{\beta_{k+1}^{HS}} + \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1}^T g_{k+1} + g_{k+1}^T s_k \\ \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} \left(\frac{y_k^T s_k}{\alpha_k y_k^T g_{k+1}} \frac{\|g_{k+1}\|_1^2}{\|g_k\|_1^2} + \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1}^T g_{k+1} + g_{k+1}^T s_k \end{aligned}$$

Using the following inequality $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$ [37], we obtain:

$$\begin{aligned} \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\left(\frac{y_k^T s_k}{y_k^T g_{k+1}} + \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1}^T g_{k+1} + g_{k+1}^T s_k \\ \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\left(\frac{y_k^T s_k}{y_k^T g_{k+1}} + \alpha_k \frac{\|g_k\|_1^2}{\sqrt{n} \|g_{k+1}\|_1^2} \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) g_{k+1}^T g_{k+1} + g_{k+1}^T s_k \\ \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\left(g_{k+1}^T g_{k+1} \frac{y_k^T s_k}{y_k^T g_{k+1}} + \alpha_k \frac{\|g_k\|_1^2}{\sqrt{n}} \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) + g_{k+1}^T s_k \\ \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\left(g_{k+1}^T g_{k+1} \frac{y_k^T s_k}{y_k^T g_{k+1}} + \alpha_k \frac{\sqrt{n} \|g_k\|_1^2}{\sqrt{n}} \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) + g_{k+1}^T s_k \\ \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\left(g_{k+1}^T g_{k+1} \frac{y_k^T s_k}{y_k^T g_{k+1}} + \alpha_k \|g_k\|_1^2 \frac{s_k^T g_{k+1}}{y_k^T g_{k+1}}\right) + g_{k+1}^T s_k \\ &\because g_{k+1}^T s_k = g_{k+1}^T s_k - g_k^T s_k + g_k^T s_k = y_k^T s_k + g_k^T s_k < y_k^T s_k \\ \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\frac{1}{y_k^T g_{k+1}} \left(g_{k+1}^T g_{k+1} y_k^T s_k + \alpha_k \|g_k\|_1^2 s_k^T y_k\right) + s_k^T y_k \\ \alpha_k \frac{\|g_k\|_1^2}{\|g_{k+1}\|_1^2} g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \left(g_{k+1}^T g_{k+1} + \alpha_k \|g_k\|_1^2 - y_k^T g_{k+1}\right) \\ g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \left(g_{k+1}^T g_{k+1} + \alpha_k \|g_k\|_1^2 - y_k^T g_{k+1}\right) \frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2} \\ g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \left(g_{k+1}^T g_{k+1} + \alpha_k \|g_k\|_1^2\right) \frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2} \\ g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \left(g_{k+1}^T g_{k+1} \frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2 + \alpha_k \|g_k\|_1^2} \frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2}\right) \\ g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \left(g_{k+1}^T g_{k+1} \frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2} + \alpha_k \|g_k\|_1^2 \frac{\|g_{k+1}\|_1^2}{\alpha_k \sqrt{n} \|g_k\|_1^2}\right) \\ g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \left(g_{k+1}^T g_{k+1} \frac{\|g_{k+1}\|_1^2}{\alpha_k \|g_k\|_1^2} + \frac{\|g_{k+1}\|_1^2}{\sqrt{n}}\right) \end{aligned}$$

$$g_{k+1}^T d_{k+1} \leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \|g_{k+1}\|_1^2 \left(\frac{g_{k+1}^T g_k}{\alpha_k \|g_k\|_1^2} + \frac{1}{\sqrt{n}} \right)$$

If we use the Cuchy–Schwarz inequality, then:

$$\begin{aligned} g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{y_k^T g_{k+1}} \|g_{k+1}\|_1^2 \left(\frac{g_{k+1}^T g_k}{\alpha_k \|g_k\|_1^2} + \frac{1}{\sqrt{n}} \right) = -\frac{s_k^T y_k}{y_k^T g_{k+1}} \|g_{k+1}\|_1^2 \left(\frac{\beta_{k+1}^{FR}}{\alpha_k \sqrt{n}} + \frac{1}{\sqrt{n}} \right) \\ g_{k+1}^T d_{k+1} &\leq -\frac{s_k^T y_k}{\sqrt{n} y_k^T g_{k+1}} \|g_{k+1}\|_1^2 \left(\frac{\beta_{k+1}^{FR}}{\alpha_k} + 1 \right). \end{aligned}$$

by assumption $y_{k+1}^T g_{k+1} > 0$ Wolfe condition and the by $s_k^T y_k > 0$

$$\therefore g_{k+1}^T d_{k+1} < 0$$

The proof is complete.

5. COMPUTATIONAL RESULTS AND COMPARISONS

This part presents the performance of MATLAB 2020 implementation with HP laptop 4 GB RAM and 500 GB hard drive of the proposed SKH-CG algorithm on a set of fuzzy nonlinear equations taken from [6]. We compared the performance of this algorithm with the Fletcher-Reeves (FR), Hisham- Khalil (KH) algorithms. The algorithms were compared in terms of the number of iterations, optimal value of the function and optimal value of the variables, as shown in Table (1). These algorithms are implemented with standard Wolfe conditions, in which $\sigma = 0.1$ and $\sigma = 0.11$, where the initial step-size $\alpha = \frac{1}{\|g_k\|}$ and initial guess for other iterations. That is, $(k > 1)$ is $\alpha_k = \alpha_{k-1} \frac{\|d_{k-1}\|}{\|g_k\|}$. In the all cases, the stopping criterion is $\|g_{k+1}\| < 10^{-6}$ and the maximum number of iteration is 2000. Our comparison includes the following aspects:

- 1- It:- number of iterations;
- 2- x-best:- optimal variable and
- 3- f-best:- optimal function value.

Table (1) shows the details of the results for the (SKH-CG) algorithm versus the FR-CG algorithm. The numerical solutions are also plotted in Figs. (1), (2) and (3).

Example 1. Consider the following fuzzy nonlinear equation [6]:

$$(3, 4, 5)x^2 + (1, 2, 3)x = (1, 2, 3).$$

Without any loss of generality, assume that x is positive. The parametric form of this equation is as follows:

$$\begin{cases} (3+r)x^2(r) + (1+r)\underline{x}(r) - (1+r) = 0, \\ (5-r)\bar{x}^2(r) + (3-r)\bar{x}(r) - (3-r) = 0. \end{cases}$$

The preceding system needs the following initial values: for $r = 1$

$$\begin{cases} 4x^2(1) + 2\underline{x}(1) - 2 = 0, \\ 4\bar{x}^2(1) + 2\bar{x}(1) - 2 = 0, \end{cases}$$

for $r = 0$

$$\begin{cases} 3x^2(0) + \underline{x}(0) - 1 = 0, \\ 5\bar{x}^2(0) + \bar{x}(0) - 3 = 0, \end{cases}$$

with initial values

$$x_0 = (\underline{x}(0), \underline{x}(1), \bar{x}(1), \bar{x}(0)) = (0.434, 0.5, 0.5, 0.681).$$

Example 2. Consider the following fuzzy nonlinear equation [6]:

$$(4, 6, 8)x^2 + (2, 3, 4)x - (8, 12, 16) = (5, 6, 7).$$

Without any loss of generality, assume that x is positive. The parametric form of this equation is as follows:

$$\begin{cases} (4 + 2r)\underline{x}^2(r) + (2 + r)\underline{x}(r) - (3 + 3r) = 0, \\ (8 - 2r)\bar{x}^2(r) + (4 - r)\bar{x}(r) - (9 - 3r) = 0. \end{cases}$$

The preceding system needs the following initial values: for $r = 1$

$$\begin{cases} 6x^2(1) + 3x(1) - 6 = 0, \\ 6\bar{x}^2(1) + 3\bar{x}(1) - 11 = 0, \end{cases}$$

for $r = 0$

$$\begin{cases} 4x^2(0) + 2x(0) - 3 = 0, \\ 8\bar{x}^2(0) + 4\bar{x}(0) - 9 = 0, \end{cases}$$

with initial values

$$x_0 = (\underline{x}(0), \underline{x}(1), \bar{x}(1), \bar{x}(0)) = (0.651, 0.7808, 0.7808, 0.8397).$$

Example 3. Consider the following fuzzy nonlinear equation [6]:

$$(1, 2, 3)x^3 + (2, 3, 4)x^2 + (3, 4, 5) = (5, 8, 13).$$

Without any loss of generality, assume that x is positive. The parametric form of this equation is as follows:

$$\begin{cases} (1 + r)\underline{x}^3(r) + (2 + r)\underline{x}^2(r) - (2 + 2r) = 0, \\ (3 - r)\bar{x}^3(r) + (4 - r)\bar{x}^2(r) - (8 - 4r) = 0. \end{cases}$$

The preceding system needs the following initial values: for $r = 1$

$$\begin{cases} 2\underline{x}^2(1) + 3\underline{x}(1) - 4 = 0, \\ 2\bar{x}^2(1) + 3\bar{x}(1) - 4 = 0, \end{cases}$$

for $r = 0$

$$\begin{cases} \underline{x}^2(0) + 2x(0) - 2 = 0, \\ 3\bar{x}^2(0) + 4\bar{x}(0) - 8 = 0, \end{cases}$$

with initial values

$$x_0 = (\underline{x}(0), \underline{x}(1), \bar{x}(1), \bar{x}(0)) = (0.76, 0.91, 0.91, 1.06).$$

Table 1. Numerical results for the preceding examples

Examples	FR ALGORITHM			KH ALGORITHM			SKH-CG ALGORITHM		
	It	x-best	f-best	It	x-best	f-best	It	x-best	f-best
1	8	0.4343 0.5000 0.5000 0.5307	5.2505e- 014	6	0.4343 0.5000 0.5000 0.5307	1.7016e- 011	7	0.4343 0.5000 0.5000 0.5307	2.2370e- 014
2	12	0.6514 0.7808 0.7808 0.8397	2.3998e- 010	9	0.6514 0.7808 0.7808 0.8397	1.9513e- 011	9	0.6514 0.7808 0.7808 0.8397	4.7224e- 011
3	19	0.8393 0.9108 0.9108 1.0564	2.6011e- 011	13	0.8393 0.9108 0.9108 1.0564	1.6756e- 013	10	0.8393 0.9108 0.9108 1.0564	7.7624e- 012

FIGURE 1. Drawing Solution to Example 1 using the SKH-CG Algorithm

FIGURE 2. Drawing Solution to Example 2 using the SKH-CG Algorithm

FIGURE 3. Drawing Solution to Example 3 using the SKH-CG Algorithm

6. CONCLUSIONS

The main purpose of this study was to apply a spectral CG algorithm that can be used to solve fuzzy nonlinear equations as an alternative to the typical analytical technique. A fuzzy nonlinear problem was converted to parametric form and solved thereafter using spectral CG algorithms. The numerical findings showed that the spectral Hisham- Khalil (SKH-CG) algorithm works substantially in all tested problems solved. This proposed algorithm can also be used in other fields, such as artificial neural networks, fuzzy neural networks and swarming algorithms, as well as in solving optimization problems.

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CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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