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Rad-Quasi-Prime Submodules

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Abstract
Consider a left J-module I. The present study introduces the concept of rad-quasi-prime submodule that serves as a dual popularization of both quasi-prime submodules and primary submodules. An apposite submodule A of a J-module I named as rad-quasi-prime if for all $u \in I$ and $a, b \in J$ with $abu \in A$ satisfies either $au \in A$ or $bu \in rad(A)$. Numerous facts and characterizations regarding this concern are acquired.

Key Words: Rad-quasi-prime submodule, quasi-prime sub modules, primary sub modules, prime sub modules.

1. INTRODUCTION

This study aims to achieve its objectives through the execution of the proposed research. In mathematical notation, the symbol “J” is used to represent a ring with an identity. Similarly, the symbol “I” is used to denote a left J module.

We use notations ⊆ to denote inclusion. For a submodule A of a J-module I, we let [A:_R I] indicate the ideal ={r∈R,rI⊆A}. A proper sub modules A of an J- module I named as prime denoted by A≤p I if for all m∈I and a∈J with am∈A implies m∈A or a∈[A:_R I]. The radical for a sub module A, which is symbolized by rad(A)={⋂_(A⊆B)▒B, where B≤p I }. If A isn’t in any prime, the radical of A can be denoted as rad(A)=I. Furthermore, it can be stated that a proper submodule A is considered to be a radical sub module of I if A=rad(A)[1]. A proper sub module of an J-module I A named as primary denoted by A≤pr I if for each m∈I and a∈J with am∈A implies u∈A or a^n∈[A:I] (that is, a∈√([A:I])) [2]. Further, in Z as Z-module. The submodule p^n Z is a primary sub module if and only if p is a prime number and n is a positive integer. A proper sub module A is named as semiprime if for each u∈I and a∈J with a^2 u∈A infers au∈A [3]. An apposite sub module A of an J-module I named as quasi-prime (QP) denoted by A≤QP I if for all u∈I and a,b∈J with abu∈A implies that either au∈A or bu∈A [4]. A proper sub module A of an J- module I named as 2- absorbing if for all u∈I and a,b∈J with abu∈A indicates that either au∈A or bu∈A or ab∈[A:I] [5]. A proper submodule A is called 2- absorbing primary if for all u∈I and a,b∈J with abu∈A implies that either au∈rad(A) or bu∈rad(A) or ∈[A:I] [6]. It is obvious that each 2-absorbing sub module is 2-absorbing primary.

This work consists of two sections. In section 2, we represent the concept of rad-QP submodules where facts and characterizations about this type of submodules are discussed.

We would like to record that for any prime number p, pZ⨁Z, Z⨁pZ are prime sub-modules of Z⨁Z as Z-module. It can be observed that in the case of non-equal prime numbers p and q, pZ⨁qZ cannot be classified as a prime sub module of Z (suppose the values p=2, q=3).

The subsequent discussion employs the notation ℤ, 〖Z〗_(p^∞) and 〖Z〗_n=Z/nZ to refer to integers the p-Prüfer group and the residue ring of integers modulo n, respectively.

2. RAD-QUASI-PRIME SUBMODULES

Definition 2.1

A proper sub module C of a J-module I named as rad-QP, represented as (C Rad_QP of *I*) if for each $n \in I$ and $p, q \in J$ with $pqn \in C$ implies that either $pn \in C$ or $qn \in rad(C)$.

Proposition 2.2

For every submodule $A \leq_{QP} I$, then *I* is a rad-QP.

Proof

Let $A \leq_{QP} I$ and $abu \in A$ for some $s, t \in J, u \in I$. Hence, either $su \in A$ or $tu \in A \subseteq rad(A)$ is desired.

The opposite of Proposition (2.2) does not hold universally, as demonstrated in the next example:

Example 2.3

Let $I = \mathbb{Z}$ as $\mathbb{Z}$- module and $A = 4\mathbb{Z}$ be a submodule of I where $rad(A) = 2\mathbb{Z}$. By a simple calculation, we see that is A is a rad-QP. But A is not QP since taking $s = 2, t = 2, u = 1$, then $stu \in A$ while $su \notin A$ or $tu \notin A$ does not make A a QP.

Theorem 2.4

Let I be a J-module and A be submodule of I such that $rad(A) \neq I$. The following statements are of equal meaning:

1) A Rad_QP of *I*
2) $rad(A) \leq_{QP} I$

Proof

(1) ⇒ (2) Assuming that A Rad_QP of I and $sbu \in rad(A)$ for some $s, b \in J$ and $u \in I$, then either $su \in rad(A)$ or $bu \in rad\big(rad(A)\big) = rad(A)$ as desired.

(2) ⇒ (1) Let $abu \in A$ for some $s, b \in R$ and $u \in I$. Put that $su \notin A$, we must prove that $bu \in rad(A)$. Since $sbu \in A \subseteq rad(A)$, so $sbu \in rad(A)$ and by hypothesis, $rad(A)$ is QP implies that either $su \in rad(A)$ or $bu \in rad(A)$ as wanted.

Remarks and Examples 2.5

1) Each one of the semiprime and rad-QP submodules does not lead to the other, like the following example: $6\mathbb{Z}$ is a semi-prime sub module of $\mathbb{Z}$ as $\mathbb{Z}$ - module because $6\mathbb{Z} = rad(6\mathbb{Z}) = \sqrt{6\mathbb{Z}}$, while $6\mathbb{Z}$ is not rad-QP because $2.3.1 \in 6\mathbb{Z}$, while $2.1 \notin rad(6\mathbb{Z})$ and $3.1 \notin rad(6\mathbb{Z})$. On the other side, $4\mathbb{Z}$ is rad-QP, but it is not semiprime since $2^2.1 \in 4\mathbb{Z}$ and $2.1 \notin 4\mathbb{Z}$.
2) Clearly, if A is a radical submodule (that is, $d(A) = A$) of a J-module I, then the concept rad-QP and QP are equivalent.
3) For any J-module I, if $rad(A) = I$ then A Rad_QP of *I*.
4) Every submodule of the $\mathbb{Z}$-module, $\mathbb{Z}_{p^\infty}$ can be written as $A =< \frac{1}{p^n} + \mathbb{Z} >$, where $= 0,1,2, \ldots$. It's well known that every submodule of $\mathbb{Z}_{p^\infty}$ is not primary and, hence, not prime so that $rad(A) = \mathbb{Z}_{p^\infty}$ for every submodule A of $\mathbb{Z}_{p^\infty}$, it follows that A Rad_QP of $\mathbb{Z}_{p^\infty}$ by (3).
5) Every sub module of $\mathbb{Q}$ as $\mathbb{Z}$- module is rad-QP because the zero submodule is the only prime submodule in $\mathbb{Q}$. Hence, $rad(A) = \mathbb{Q}$ for all non- zero sub module A of $\mathbb{Q}$ so by (3), A is rad-QP. If $A =< 0 >$ then $rad(A) =< 0 > \leq_P \mathbb{Q}$, and with theorem (2.4), A is rad-QP.
6) If A is a direct summand of a submodule B of a J- module I then A may not be rad-QP. For example, $A =< \bar{6} >$ is a direct summand submodule of $B =< \bar{2} >$ in $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module where $< \bar{6} > \bigoplus < \bar{4} > =< \bar{2} >$. But $< \bar{6} >$ isn't rad-Q P because $2.3.\bar{1} \in < \bar{6} >$ while $2.\bar{1} \notin rad(< \bar{6} >) =< \bar{6} >$ and $3.\bar{1} \notin rad(< \bar{6} >)$.
7) Let $H \subseteq B$ be sub modules of a J-module I. If B Rad_QP of I, then H need not. For instance, $H =< \bar{6} > \subseteq B =< \bar{2} >$ in $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module where $B \leq_P$ I, and hence, it is rad-QP while H is not rad-QP as shown in (6).

Proposition 2.6

Let I be a J- module and A be sub module of I. The next statements are equivalent:

1) A Rad_QP of I.
2) $[rad(A):<m>]$ is a prime ideal (p.id.) of J for all $m \in I$.

Proof

(1) ⇒ (2) Let a , $b \in R$ with $ab \in [rad(A):_R <m>]$. Hence $ab \in <m> \subseteq rad(A)$, thus $abm \in rad(A)$. By assumption, A Rad_QP of I and so $am \in rad(A)$ or $bm \in rad(A)$. This indicates that either $<am> \in rad(A)$ or $<bm> \in rad(A)$. It's easy to notice that $a<m>=<am>$ and $b<m>=<bm>$ so we have either $a<m> \in rad(A)$ or $b<m> \in rad(A)$, it follows that $a \in [rad(A):_R <m>]$ or $b \in [rad(A):_R <m>]$ means that $[rad(A):<m>]$ is a p.id. of J.

(2) ⇒ (1) Let $abm \in A$ where $a, b \in R$ and $m \in M$. Consider that $am \notin A$, we need verify that $bm \in rad(A)$. As $abm \in A \subseteq rad(A)$ point toward that $ab \in [rad(A):m] = [rad(A):<m>]$. By suggestion, $[rad(A):<m>]$ is a p.id. of J, that's, $a \in [rad(A):<m>]$ or $b \in [rad(A):<m>]$. So $am \in rad(A)$ or $bm \in rad(A)$ and thus A Rad_QP of I.

Theorem 2.7

The next statements are equivalent :

1) A Rad_QP of I.
2) $[rad(A):B]$ is a p.id. of J for any sub module B of I.
3) $[rad(A):<am>] = [rad(A):<m>]$ for any $m \in I$, for any $a \in J$ with $a \notin [rad(A):<m>]$.

Proof

(1) ⇒ (2) Assuming that A Rad_QP of I, $[rad(A):<m>]$ is a p.id. of J for each $m \in I$. We claim that for any submodule B of I, we have to show that $[rad(A):B]$ is a p.id. of J. Let $ab \in [rad(A):B]$. Assuming that $a \notin [rad(A):B]$ and $b \notin [rad(A):B]$, it follows that $aB \nsubseteq rad(A)$ and $bB \nsubseteq rad(A)$. Its meaning is that there is $m, n \in B$ where $am \notin rad(A)$ and $bn \nsubseteq rad(A)$, which is an opposite with assumption. Therefore, $[rad(A):B]$ is a p.id. of J for any sub module B of I.

(2) ⇒ (3) Clearly, $[rad(A):<m>] \subseteq [rad(A):<am>]$. Let $b \in [rad(A):<am>]$ with $a \notin [rad(A):<m>]$ implying that $b<am> \subseteq rad(A)$. It is easy see that $b<am> = ba<m>$ and, thus, $ba<m> \subseteq rad(A)$—that is, $ab \in [rad(A):<m>]$. By hypothesis, $[rad(A):<m>]$ is a p.id. of J and because $a \notin [rad(A):<m>]$ implies that $b \in [rad(A):<m>]$ —that is, $[rad(A):<am>] \subseteq [rad(A):<m>]$.

(3) ⇒ (1) Let $ab \in A$, where $a, b \in J$ and $m \in I$. Assume that $am \notin rad(A)$ implies $a \notin [rad(A):<m>]$, and so $[rad(A):<am>] = [rad(A):<m>]$. Furthermore, $b \in [rad(A):<am>]$; thus, $b \in [rad(A):<m>]$ implies that $bm \in rad(A)$ as desired.

Corollary 2.8

Let A Rad_QP of I, then $[rad(A):I]$ is a p.id. of J.

Proof

Directly by Theorem (2.7):

The reverse of Corollary (2 .8) isn't hold in general. For example, $A = 6\mathbb{Z} \oplus <0>$ isn't Rad_QP submodule of $I = \mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$ -module since $2.3.(1,0) \in A$ and $2.(1,0) = (2,0) \notin rad(A) = A$, nor $3.(1,0) = (3,0) \notin rad(A) = A$ while $[rad(A):I] =<0>$ is a p.id. of $\mathbb{Z}$.

Proposition 2.9

If A Rad_QP of I, then $[rad(A):aI] = [rad(A):I]$ for each $a \notin [rad(A):I]$.

Proof

Let $b \in [rad(A): aI]$ implies that $abI \subseteq rad(A)$. Then $ab \in [rad(A): I]$ and since A Rad_QP of I so $[rad(A): I]$ is a p.id. of J by Theorem (2.7), and as a result, $b \in [rad(A): I]$ because $a \notin [rad(A): I]$. It gives that $[rad(A): aI] \subseteq [rad(A): I]$. Let $b \in [rad(A): I]$ implies that $bI \subseteq rad(A)$, and so $abI \subseteq rad(A)$. Hence, $b \in [rad(A): aI]$. That is, $[rad(A): aI] = [rad(A): I]$ for each $a \notin [rad(A): I]$. The opposite of the corollary isn't hold in general. For example, $A = 6\mathbb{Z} \bigoplus < 0 >$ isn't Rad_QP of $I = \mathbb{Z} \bigoplus \mathbb{Z}$ as $\mathbb{Z}$-module while $[rad(A): I] = [rad(A): aI] =< 0 >$ for each $a \notin [rad(A): I]$.

Lemma 2.10 [7]

Let I be a finitely generated J-module. Then $\sqrt{[A: I]} = [rad(A): I]$ for each submodule A of I.

Corollary 2.11

If A Rad_QP of a finitely generated J-module I, then $[rad(A): aI] = \sqrt{[A: I]}$ for each $a \notin [rad(A): I]$.

Proof

Obviously, by Proposition (2 .9) and Lemma (2 .10).

Lemma 2.12 [8]

The intersection of any couple of different prime submodules of a J - module I is 2 - absorbing.

Proposition 2.13

Clearly, every rad-QP submodule is 2 - absorbing primary sub module.

Remark 2.14

The opposite of Propositon (2.13) is not hold generally. For example, consider $\mathbb{Z}_6$ as $\mathbb{Z}$-module, $rad(< \bar{0} >) =< \bar{2} > \cap < \bar{3} > =< \bar{0} >$ implies $< \bar{0} >$ is 2-absorbing submodule, and therefore, it is a 2-absorbing primary. But $< \bar{0} >$ isn't rad -QP sub module because $2.3.\bar{1} = \bar{0} \in < \bar{0} >$ while $2.\bar{1} \notin rad(< \bar{0} >)$ and $3.\bar{1} \notin rad(< \bar{0} >)$.

The following diagram is obvious:

Prime submodules ⇒ quasi-prime ⇒ rad-quasi-prime ⇒ 2-absorbing primary

But the reverse of these implications is not hold generally.

Lemma 2.15

Let I be a J- module, then $\sqrt{[A: I]} \subseteq [rad(A): I]$ for each submodule A of I.

Proof

If $rad(A) = I$, so the relation $\sqrt{[A: I]} \subseteq [rad(A): I]$ is hold. In case $rad(A) \neq I$, let $B \leq_p I$ with $A \subseteq B$. Hence, $[A: I] \subseteq [B: I]$ and $[B: I]$ is a p.id. of R and so $\sqrt{[A: I]} \subseteq \sqrt{[B: I]} = [B: I]$. Thus, $\sqrt{[A : I]}\, I \subseteq [B: I]I \subseteq B$; it follows that $\sqrt{[A: I]}\, I \subseteq B$; this is true for all prime submodule B containing A of I. Therefore, $\sqrt{[A:_R I]}\, I \subseteq \cap B = rad(A)$ as desired.

Proposition 2.16

Every primary submodule is rad-QP.

Proof

Let $A \leq_{pr} I$ and $abm \in A$, where $a, b \in R$, $m \in I$. Put $x = bm$, so either $x \in A$ or $a \in \sqrt{[A:_R I]} \subseteq [rad(A):_R I]$ by Lemma (2.15), and thus, $aI \subseteq rad(A)$ means $am \in rad(A)$ for each $m \in M$. Therefore, either $bm \in A$ or $am \in rad(A)$; that is, A Rad_QP of I.

Remarks and Examples 2.17

1) The converse of Proposition (2.16) isn't hold generally. Like the example, $rad(4\mathbb{Z} \bigoplus < 0 >) = 2\mathbb{Z} \bigoplus < 0 >$ is Rad_QP of $\mathbb{Z} \bigoplus \mathbb{Z}$ as $\mathbb{Z}$ - module so by Theorem (2.4), $4\mathbb{Z} \bigoplus < 0 >$ is a rad-QP submodule, while $4\mathbb{Z} \bigoplus < 0 >$ is not primary submodule since $2.(\bar{2}, \bar{0}) \in 4\mathbb{Z} \bigoplus < 0 >$, while $(\bar{2}, \bar{0}) \notin 4\mathbb{Z} \bigoplus < 0 >$ and $2(\mathbb{Z} \bigoplus \mathbb{Z}) \nsubseteq 4\mathbb{Z} \bigoplus < 0 >$.
2) In the Z-module Z, the primary and rad-QP submodules concepts are equivalent.

Proof

Let A be a submodule of $\mathbb{Z}$, implying that $A = t\mathbb{Z}$ for some positive integer t. If t is a prime number, then A is a prime submodule, and so there is nothing to prove. Let $1 \neq t$ be not a prime number. By the factorization theorem, we can write $t = {p_1}^{k_1}.{p_2}^{k_2} \dots {p_r}^{k_r}$ as a factorization of the positive integers into distinct primes P_i and k_i is integers where $i = 1,2, \dots . r$. Thus, $\sqrt{A} = \sqrt{t\mathbb{Z}}$= $\sqrt{< {p_1}^{k_1}.{p_2}^{k_2} \dots {p_r}^{k_r} >} = \ < p_1.p_2 \dots p_r >$. In case $r = 1$, $\sqrt{A} = \sqrt{t\mathbb{Z}} = \sqrt{< p^k >}$=$< p >$ is a prime sub module (and therefore, quasi-prime), for some a positive integer k, so by Theorem (2.4) A is rad-QP, and at the same time, A is a primary sub module since it is of the form $< p^k >$. In case $r \geq 2$ implies that A is not rad-QP, to show this, if $r = 2$, then $\sqrt{< {p_1}^{k_1}.{p_2}^{k_2} >} = < p_1.p_2 >$ is not rad-QP because $p_1.p_2.1 \in < p_1.p_2 >$, but $p_1.1 \notin < p_1.p_2 >$ and $p_2.1 \notin < p_1.p_2 >$, and hence, $A = \sqrt{< {p_1}^{k_1}.{p_2}^{k_2} >} = \ < p_1.p_2 >$ is not primary submodule. By induction $\sqrt{A} = \sqrt{t\mathbb{Z}}$= $\sqrt{< {p_1}^{k_1}.{p_2}^{k_2} \dots {p_r}^{k_r} >} = \ < p_1.p_2 \dots p_r >$, for each $r \geq 2$ that is A isn't rad-QP, and hence, it is a primary submodule.

3) Primary submodules and QP submodules are independent. For example, $4\mathbb{Z} \leq_{pr} \mathbb{Z}$ as Z-module while $4\mathbb{Z}$ isn't QP because $2.2.1 \in 4\mathbb{Z}$ and $2.1 \notin 4\mathbb{Z}$. On the other hand, $2\mathbb{Z} \oplus < 0 > \leq_{QP} \mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module, but it isn't primary since $2(3\ 0) = (6,0) \in 2\mathbb{Z} \oplus < 0 >$, while $(3\ 0) \notin 2\mathbb{Z} \oplus < 0 >$ and $2(\mathbb{Z} \oplus \mathbb{Z}) = 2\mathbb{Z} \oplus 2\mathbb{Z} \nsubseteq 2\mathbb{Z} \oplus < 0 >$.

Lemma 2.18 [9]

Let I be a J-module such that $I = \oplus_{\alpha} I_{\alpha}$ is a direct sum of submodules I_{α} $(\alpha \in \Lambda)$. For all $\alpha \in \Lambda$, take A_{α} as a submodule of I_{α} and let $A = \oplus_{\alpha} A_{\alpha}$. Then $rad(A) = \oplus_{\alpha \in \Lambda} rad(A_{\alpha})$.

Remark 2.19

The direct sum of rad-QP submodules may be not rad-QP. For example, $8\mathbb{Z}$ and $9\mathbb{Z}$ are rad-QP sub modules of $\mathbb{Z}$ as $\mathbb{Z}$- module, while $8\mathbb{Z} \oplus 9\mathbb{Z}$ isn't rad_QP submodule of $\mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$ -module since $2.3.(1,1) \in 8\mathbb{Z} \oplus 9\mathbb{Z}$ but $2.(1,1) \notin rad(8\mathbb{Z} \oplus 9\mathbb{Z}) = rad(8\mathbb{Z}) \oplus rad(9\mathbb{Z}) = 2\mathbb{Z} \oplus 3\mathbb{Z}$ by Lemma (2.18), and $3.(1,1) \notin rad(8\mathbb{Z} \oplus 9\mathbb{Z}) = 2\mathbb{Z} \oplus 3\mathbb{Z}$.

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